Bounding options prices using SDP with change of numeraire

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Outline

- Problem description
- SDP representations and approximations
- Applications: options pricing with change of numeraire
- Numerical results

Problem description

The central problem of the study is:

$$
\begin{aligned}
&& \inf_{\mu} \text{ or } \sup_{\mu} \int_S f(x)d\mu \\
(P) && \text{s.t.} \int_S x^\alpha d\mu = \sigma_\alpha, \quad \forall \alpha \in \mathbb{I}_d \\
&& \mu(x) \in M(S),
\end{aligned}
$$

where $f(x) : S \rightarrow \mathbb{R}$ is a real-valued measurable function on S. $\mathcal{B}$ is the Borel σ-field of $\mathbb{R}^n$, $S \in \mathcal{B}$ is the domain under consideration, and $M(S)$ denotes the set of finite positive Borel measures supported by S. $\mathbb{I}_d$ is a finite set defined by

Problem description

Using the upper bound case for illustration, we have:

$$(P) \qquad \begin{aligned} \sup_{\mu} & \int_S f(x) d\mu \\ \text{s.t.} & \int_S x^\alpha d\mu = \sigma_\alpha, \quad \forall \alpha \in \mathbb{I}_d \\ & \mu(x) \in M(S). \end{aligned}$$

Its dual is:

$$(D) \qquad \begin{aligned} \inf & \sum_{\alpha \in \mathbb{I}_d} \sigma_\alpha \theta_\alpha \\ \text{s.t.} & \sum_{\alpha \in \mathbb{I}_d} \theta_\alpha x^\alpha \geq f(x) \quad \forall x \in S. \end{aligned}$$

Problem description

Questions raised: What are the strong duality conditions? How do these formulations relate to conic programs?

$$
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& \mu(x) \in M(S).
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\end{aligned}
$$

Problem description

Two well-studied classes of cones, namely, the moment cones and the cones of positive semidefinite polynomials provide the answer.
The cone of moments supported in S is defined as

$$M_{n,d}(S) = \{y \in \mathbb{R}^{\mathbb{I}_d} : y_\alpha = \mathbb{E}_\mu(x^\alpha), \forall \alpha \in \mathbb{I}_d \text{ for some } \mu \in M(S)\}. \tag{1}$$

The cone of positive semidefinite polynomial is defined as

$$P_{n,d}(S) = \{\theta = (\theta_\alpha)_{\alpha \in \mathbb{I}_d} \in \mathbb{R}^{\mathbb{I}_d} : \theta(x) = \sum_{\alpha \in \mathbb{I}_d} \theta_\alpha x^\alpha \geq 0 \;\; \forall x \in S\}. \tag{2}$$

The cones $M_{n,d}(S)$ and $P_{n,d}(S)$ are related through duality, which is

$$P_{n,d}(S) = M_{n,d}(S)^*. \tag{3}$$

Problem description

$$z_P = \sup_{\mu} \int_S f(x) d\mu \qquad\qquad z_D = \inf \sum_{\alpha \in \mathbb{I}_d} \sigma_\alpha \theta_\alpha$$

$$(P) \quad \text{s.t.} \int_S x^\alpha d\mu = \sigma_\alpha, \quad \forall \alpha \in \mathbb{I}_d \quad (D) \quad \text{s.t.} \sum_{\alpha \in \mathbb{I}_d} \theta_\alpha x^\alpha \geq f(x) \quad \forall x \in S.$$

$$\mu(x) \in M(S).$$

Therefore, the sufficient conditions for strong duality can be summarized as follows: If either

- $\sigma \in$ Int$(M_{n,d}(S))$ or
- there exists $\theta \in \mathbb{R}^{\mathbb{I}_d}$ such that$(\theta - f) \in$ Int$(P_{n,d}(S))$.

Then $z_P = z_D$. Here, Int(S) denotes the interior of the set S.

Problem description

Moreover, when $f(x)$ is a piece-wise polynomial, the primal and dual become as follows:

$$
\begin{array}{llll}
z_{P_{pw}} = \sup \sum_{i=1}^{p} \mathbb{E}_{\mu_i}(q^i(x)) & & z_{D_{pw}} = \inf \sum_{\alpha \in \mathbb{I}_d} \sigma_\alpha \theta_\alpha & \\
(P_{pw}) \quad \text{s.t.} \sum_{i=1}^{p} \mathbb{E}_{\mu_i}(x^\alpha) = \sigma_\alpha, & \forall \alpha \in \mathbb{I}_d & (D_{pw}) \quad \text{s.t.} \sum_{\alpha \in \mathbb{I}_d} \theta_\alpha x^\alpha \geq q^i & \forall x \in S_i, i = 1, ...p, \\
\qquad \mu_i \in M(S_i), \;\; i = 1..., p, & & &
\end{array}
$$

where with disjoint Borel measurable sets $S_i \subseteq \mathbb{R}^n, \;\; i = 1, ...p$, the piece-wise polynomial $f(x) : S = \cup_{i=1}^{p} S_i \to \mathbb{R}$ is defined by $f(x) = q^i(x)$ if $x \in S_i, i = 1, ..., p$. The similar results hold: If either

- $\sigma \in$ Int$(M_{n,d}(S))$ or
- there exists $\theta \in \mathbb{R}^{\mathbb{I}_d}$ such that$((\theta - q^1)^T, ..., (\theta - q^p)^T)^T \in$ Int$(P_{n,d}(S_1) \times \cdots \times P_{n,d}(S_p))$.

Then $z_{P_{pw}} = z_{D_{pw}}$. Again, Int(S) denotes the interior of the set S.

SDP representations

There are many semidefinite representation results for both the moment cones (see e.g. Lasserre,2002) and the cones of positive semidefinite polynomials (see, e.g. Bertsimas and Popescu, 2000). Our problems can therefore be solved efficiently by SDP.

$$
\begin{aligned}
& z_{P_{pw}} = \sup \sum_{i=1}^{p} \langle q^i, y^i \rangle \\
(P_{pw}) \qquad & \text{s.t.} \sum_{i=1}^{p} y^i = \sigma, \\
& y^i \in M_{n,d}(S_i), \quad i = 1..., p.
\end{aligned}
\qquad\qquad
\begin{aligned}
& \sup \langle c, x \rangle \\
(P_{SDP}) \quad & \text{s.t.} Ax = b, \\
& x \in K,
\end{aligned}
$$

$$
\begin{aligned}
& z_{D_{pw}} = \inf \langle \sigma, \theta \rangle \\
(D_{pw}) \qquad & \text{s.t.} \theta - q^i \in P_{n,d}(S_i), \quad i = 1, ...p,
\end{aligned}
\qquad
\begin{aligned}
& \inf \langle b, y \rangle \\
(D_{SDP}) \quad & \text{s.t.} A^* y - c \in K^*,
\end{aligned}
$$

where, $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map and $\langle Ax, y \rangle = \langle x, A^* y \rangle$, K and

K^* are a closed convex cone and its dual cone, respectively. When the cone K is defined

SDP representations

- Finding SDP representations of the cones essentially is to decompose a nonnegative polynomial on $\mathbb{R}^n$ into a sum of squares of other polynomials, which is Hilbert's 17th problem presented at Paris in 1900.
  - Hilbert gave a complete description of the problem in some special cases: $n \leq 2; m = 2;$ or $n = 3, m = 4;$ where n and m denotes dimensions and moments respectively.
  - In general cases, Reznick (1995) provides a nice result showing that if $\theta(x) \in \mathbb{R}^n$ is positive definite it is always possible to write $(\sum_{i=1}^{n} x_i^2)^r \theta(x)$ as a sum of squares. A semidefinite approximation scheme for $P_{n,d}$ is therefore derived by Zuluaga (2006)
  - On the other hand, Lasserre (2002), exploiting Putinar(1993)'s theorem, also proposes a hierarchy of semidefinite approximations for the moment cones.

SDP representations

Important results from Lasserre (2006):

Theorem 1. *Given a vector* $y = (y_0, y_1, ..., y_{2d}) \in \mathbb{R}^{2d+1}$*, the following statements are true:*
(a) *With regard to the truncated Hausdorff moment problem,*

$$M_d(y) \succeq 0 \text{ and } M_d(g, y) \succeq 0,$$

with $x \to g(x) := (b - x)(x - a)$*, are necessary and sufficient conditions for the elements of* y *to be the first* $2d + 1$ *moments of a measure supported on* $[a, b]$*.*
(b) *With regard to the truncated Stieltjes moment problem,*

$$M_d(y) \succ 0 \text{ and } M_d(g, y) \succ 0,$$

with $x \to g(x) := x - a$*, are sufficient conditions for the elements of y to be the first* $2d + 1$ *moments of a measure supported on* $[a, +\infty)$*.*

The proof is provided by Curto and Fialkow (1991).

Application:options pricing

The basic formulation for the value of a European type call options written on one underlying asset X is given by

$$C_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[g(S_T)] = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+],$$

where $T > 0$ is the option's maturity time, K is the option's strike price, and $\mathbb{Q}$ is risk-neutral probability (martingale) measure. If the distribution of S_T is known, by the definition of expectation we can integrate the RHS of the equation above and achieve the option price (Black-Scholes).

Options pricing

Instead of assuming the distribution, if we only know the moments of the distribution, then the upper or lower bound of the price can be formulated as follows:

$$\max_{\mathbb{Q}} \text{ or } \min_{\mathbb{Q}} \; \mathbb{E}_{\mathbb{Q}}(f(x)) = \mathbb{E}_{\mathbb{Q}}(\max\{S_T - K, 0\})$$
$$\text{s.t.} \mathbb{E}_{\mathbb{Q}}(x^{\alpha}) = \sigma_{\alpha}, \quad \alpha \in \mathbb{I}_d$$
$$\mathbb{Q}(x) \in M(S).$$

As can be seen that if we treat the max function as a piecewise polynomial, from previous results by letting $p = 2, q^1(x) = x - E$,$q^2(x) = 0$, and $S = \mathbb{R}^+(S_1 = [K, +\infty), S_2 = [-\infty, K])$.

Options pricing

By employing the previous results, we therefore derive the following formulations for the upper bound problem:

$$
\begin{aligned}
& z_{P_{pw}} = \sup \sum_{i=1}^{p} \langle q^i, y^i \rangle \\
(P_{pw}) \qquad & \text{s.t.} \sum_{i=1}^{p} y^i = \sigma, \\
& y^i \in M_{n,d}(S_i), \;\; i = 1..., p,
\end{aligned}
\qquad\qquad
\begin{aligned}
& z_{D_{pw}} = \inf \langle \sigma, \theta \rangle \\
(D_{pw}) \qquad & \text{s.t.} \theta - q^i \in P_{n,d}(S_i), \quad i = 1, ...p,
\end{aligned}
$$

where in this case $p = 2, q^1(x) = x - K$,$q^2(x) = 0$, and $S_1 \cup S_2 = S = \mathbb{R}^+$. Furthermore,

it is easy to see that the more moments (σ)we know the better bounds we can achieve.

The convergence of the bounds to the exact value is, however, only guaranteed when the

probability measure is moment determinate(see e.g. Lasserre(2006)).

Options pricing with change of numeraire

Problem: when we increase the number of moments, we often run into numerical problems! why? By Looking at the piece q^1 the supported region is $[K, +\infty]$, and therefore according to Theorem 1 only positive definite moment and localizing matrices are sufficient to guarantee the sequences in the matrices are indeed the moments of the measure.
However, for the European call option we observe that

$$C_0 = S_0 \mathbb{E}^{\mathbb{Q}}(1 - \frac{K}{S_T})^+,$$

if we take the discounted stock out of the expectation and utilize the martingale property. The resulting upper bound problem becomes calculating the upper bound of the martingale measure supported on $[0, 1]$.

Options pricing with change of numeraire

The experiment is to calculate the tight bounds on European style exchange options which defined as

$$C_0 = \mathbb{E}^{\mathbb{Q}}[(S_T^1 - S_T^2)^+].$$

Bounding the no arbitrage price of exchange options, at first glance, has two dimensions, we can have two pieces of the support region of the measure, one is $S_T^1 - S_T^2 \geq 0$ and the other is $S_T^1 - S_T^2 \leq 0$ on $\mathbb{R}^2$. Zuluaga and Pe$\tilde{n}$a have computed the upper bound with first two moments of the measure supported on $\mathbb{R}_+^2$.
However, we note that this problem can be simplified to one dimension problem by using the change of numeraire. In fact, we actually have two numeraires to choose S_t^1 or S_t^2, but choosing S_t^1 as the numeraire, we will have bounded support region for the objective measure:

$$C_0 = S_0^1 \mathbb{E}^{\mathbb{Q}}[(1 - \frac{S_T^2}{S_T^1})^+], \tag{4}$$

where $dY_t' = Y_t'(\sigma_2 - \sigma_1)dB_t^{\mathbb{Q}'}, \;\; Y_t' = \frac{S_t^2}{S_t^1}$.

Hedging strategies

With regard to hedging strategies, we notice that a hedging strategy can be calculated via the dual formulation. The constraints of the dual formulation (D) show

$$\sum_{\alpha \in \mathbb{I}_d} \theta_\alpha x^\alpha \geq f(x).$$

If we take expectations on both sides, we obtain

$$\sum_{\alpha \in \mathbb{I}_d} \theta_\alpha \sigma_\alpha \geq \mathbb{E}[f(x)],$$

which give us an over-hedged strategy with θ_0 in the cash bond and $\sum_{\alpha \in \mathbb{I}_d} \theta_\alpha \sigma_\alpha (\alpha \neq 0)$ in risky assets.

Numerical results

We compute the upper and lower bounds of a European call option with knowing up to fourth moments of the martingale measure. For convenience of comparison, we assume the discounted stock is an exponential martingale as in the Black-Schole model with inputs such as

$$S_0 = 40, \;\; r = 0.06, \;\; \sigma = 0.2, \;\; T = 1/52.$$

	4-moments	3-moments	2-moments	
Strike	[UB, LB]	[UB, LB]	[UB, LB]	BS
30	10.0347 10.0346	10.0453 10.0346	10.0518 10.0346	10.0346
35	5.0419 5.0404	5.0768 5.0404	5.0866 5.0404	5.0404
40	0.5777 0.3422	0.5777 0.0461	0.5777 0.0461	0.4658
45	0.0042 0.0000	0.0773 0.0000	0.0773 0.0000	0.0000
50	0.0008 0.0000	0.0480 0.0000	0.0480 0.0000	0.0000

Numerical results

We show the upper bounds caculated with change of numeraire and with up to fourth moments.

		Zuluaga et al (2005)	$\frac{S_T^2}{S_T^1}$ (Upper bounds)	
ρ	Exact	2-mom	2-mom	4-mom
-1.0	0.1801	0.2206	0.2242	0.2114
-0.5	0.1600	0.1958	0.1961	0.1888
0	0.1361	0.1660	0.1641	0.1621
0.5	0.1051	0.1268	0.1241	0.1240
1	0.0500	0.0504	0.0516	0.0502

Numerical results

In the case of multi-dimension, we compute the upper bound of an outperformance option over 3 stocks with knowing up to 4th moments of the return distributions, which extends Boyle and Lin(1997)'s results.

Strike	UB(4th moment)	UB(Boyle-Lin)	Exact
30	17.5246	19.4644	16.35
35	14.3504	15.5331	12.38
40	11.4852	11.9472	8.98
45	7.7187	8.9153	6.27
50	5.0801	6.6096	4.23

Assuming the three assets follow a correlated multivariate lognormal distribution, other parameters see Boyle and Lin(1997). Question: How to improve?

The End.