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Robust Portfolio Optimization: A Conic Programming Approach

K. Ye
P. Parpas
B. Rustem

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Kai Ye, Panos Parpas, Berç Rustem

Abstract

Markowitz's mean-variance model (MMV) quickly became very popular in theory and practice due to its simplicity and plausibility. However, there is limited applicability of this model because of its extreme sensitivity to input parameters. Inputs such as the mean and the covariance matrix of returns are only estimates of parameters of unknown probability distributions. These estimations are prone to errors. In order to cope with the effect of the estimation errors, we propose to construct a one-period robust mean-variance model by introducing uncertainty regions over the mean vector and the second moment matrix of returns. The resulting robust mean-variance portfolio selection problem can usually be cast as a conic program which can be solved efficiently. In this paper, both second order cone program (SOCP) and semidefinite program (SDP) formulations are proposed. The numerical results show the portfolios generated by the proposed robust mean-variance model outperform MMV's portfolios. Moreover, the robust efficient frontiers are presented.

1 Introduction: robust max return

The mean-variance model has been the fundamental theoretical framework for portfolio construction since Markowitz [8] introduced his Nobel prize-winning work. This model, however, has also been challenged by investment professionals from both academia and financial institutions due to its high sensitivity to the input data such as expected returns and the covariance matrix of returns. For example, Broadie [3] investigates this matter and concludes that the effect of errors in parameter estimates on the results of mean-variance analysis can be surprisingly large. Robust optimization framework has been introduced to cope with this effect by ensuring that decisions are reliable even if estimates of the input parameters are incorrect. Recently, Tütüncü and Koenig [13] assuming componentwise uncertainty sets over the mean return vector and the covariance matrix, respectively, formulate the robust portfolio selection problem as a saddle-point problem that involves semidefinite constraints. Goldfarb and Iyengar [5] also develop a robust factor model for the asset returns and cast the robust portfolio selection problems to SOCP problems which can be solved efficiently. In addition, El-Ghaoui et al. [6] also consider the robust portfolio problem via the approach of worst-case Value-at-Risk (VaR), which employs similar SDP techniques exploited in this paper.

One shortcoming of these models is that the authors seem to be considering separated uncertainty sets for the expected return vector $\hat{x}$ and for the covariance matrix Γ, which means one has inconsistent probability measures over the uncertainty set for the expected return vector and the uncertainty set for the covariance matrix. More explicitly, the worst-case variance problem in these models has different probability measures over the mean and the covariance of returns. This leads to a worse than worst-case situation and might be over-conservative. Moreover, it is too restricted to assume that returns follow normal distributions as indicated by Goldfarb and Iyengar [5], because the distribution of returns in practice is not Gaussain.

To mitigate these shortcomings, we propose a robust portfolio optimization and selection model by using conic programming approaches under the mean-variance framework which introduces uncertainty sets to the mean vector and the second moment of returns, respectively. Note that we do not have uncertainty set over covariance matrix directly, but two uncertainty sets over the mean vector and the second moment matrix of returns simultaneously under a probability measure. The robust portfolio selection problem can then be cast as a single convex SDP problem by using duality theory, or a semi-infinite conic programming problem. A salient feature is that the expected return and the covariance matrix of returns are defined in a consistent manner. In other words, we consider the worst-case variance of a portfolio under a unique probability measure. Furthermore, we do not assume any specific distributions over returns, but we do require the knowledge of the first two moments of the distribution of returns either exactly or in uncertainty sets.

The uncertainty sets in many (aforementioned) literatures are either polytopes or ellipsoids. In this paper, we mainly consider ellipsoidal uncertainty set on the second moment matrix of returns and componentwise bounds on the mean vector of returns. The ellipsoidal uncertainty over the mean vector is also studied in the introduction. In the sequel, we first only take the estimation error of the expected returns into account as an introduction, which is first introduced by Ceria and Stubbs [4]. A robust maximum return problem with given risk levels is cast as an SOCP problem. Efficient frontiers are generated for comparison and analysis between the MMV model and its robust counterpart. In the numerical results (cf.Figure1), it can be easily seen that portfolios generated by the robust mean-variance model outperform the MMV's portfolios in terms of sensitivity to the estimation error of the expected returns. Following in the similar vein, we then propose our model that considers the robust minimum variance problem in the section 2 of the paper. We start from considering the minimum variance problem with componentwise uncertainty set over the second moment matrix of returns assuming that the mean vector is known and fixed. We then study the more general case where componentwise uncertainty regions are introduced to both the expected returns (mean square matrix cf. section 2.1) and the second moment matrix of returns. The resulting robust mean-variance portfolio selection problem is converted into an SDP problem and an SOCP-SDP semi-infinite problem. In addition, the robust efficient frontier is presented.

1.1 Robust maximum return

We first focus on considering the effects of errors in estimates of expected returns in this subsection. According to Markowitz's model, the maximum return problem is a quadratic (QP) problem that can be formulated as follows:

$$\begin{aligned} \max_{w \in \mathbb{R}^n} \quad & \hat{x}^T w - \lambda w^T \Gamma w \\ \text{s.t} \quad & w \in \bar{W}, \end{aligned}$$

where $\hat{x}$ is the estimated expected returns of x, the matrix Γ is the covariance matrix of estimated returns, and $\bar{W}$ represents the set of feasible portfolios. For instance, we can define $\bar{W} = \{w \in \mathbb{R}^n \,|\, \sum_{i=1}^n w_i = 1, w \geq 0\}$. Broadie [3] shows that the effects of estimation error can be surprisingly large. In order to take the worst case into account, as proposed by Ceria and Stubbs [4], it is assumed that the vector of true expected returns $\hat{x}_t$ is normally distributed and lies in the confidence region (ellipsoid)

$$(\hat{x}_t - \hat{x}_e)^T \Gamma_e^{-1} (\hat{x}_t - \hat{x}_e) \leq k^2 \tag{1}$$

generated by estimated expected returns $\hat{x}_e$ and a covariance matrix Γ_e of the estimates of expected returns with probability η, where $k^2 = \chi_n^2(1-\eta)$ and χ_n^2 is the inverse cumulative distribution function of the chi-squared distribution with n degrees of freedom. Suppose that we always over-estimated the expected returns, the worst case of the estimated expected returns with a given portfolio can then be formulated as follows:

$$\begin{aligned} \max_{\hat{x}_e - \hat{x}_t} \quad & (\hat{x}_e - \hat{x}_t)^T \hat{w} \\ \text{s.t} \quad & (\hat{x}_e - \hat{x}_t)^T \Gamma_e^{-1} (\hat{x}_e - \hat{x}_t) \leq k^2, \end{aligned} \tag{2}$$

where $\hat{w}$ is a given portfolio. By using Lagrange multiplier method, the true expected returns of the portfolio therefore can be expressed as $\hat{x}_e\hat{w} - k||\Gamma_e^{1/2}\hat{w}||$. The problem now becomes a robust portfolio selection problem that can be formulated as follows:

$$\begin{aligned} \max_{w \in \mathbb{R}^n} \quad & \hat{x}^T w - k||\Gamma_e^{1/2} w|| \\ \text{s.t} \quad & \lambda w^T \Gamma w \leq \gamma^2 \\ & w \in \bar{W}. \end{aligned} \tag{3}$$

One can see this formulation is the traditional maximum return formulation added with an additional term $k||\hat{\Gamma}_e^{1/2}\hat{w}||$ to reduce the effect of estimation error on the optimal portfolio. This problem can be easily cast as a second order cone programming (SOCP) problem:

$$\begin{aligned} \max_{w \in \mathbb{R}^n} \quad & \hat{x}^T w - kt \\ \text{s.t} \quad & w \in \bar{W}, \\ & \gamma \geq ||\Gamma_w^{1/2}||, \\ & t \geq ||\Gamma_e^{1/2} w||, \end{aligned} \tag{4}$$

where γ is a standard deviation target and it varies when one needs to compute points on the efficient frontier. Note that the difference between

Γ and Γ_e. Γ is the covariance matrix of returns which is assumed to be known exactly at this moment. Γ_e is the covariance matrix of estimated expected returns, which is related to the estimation error arising from the process of estimating $\hat{x}$, the vector of expected returns. As can be seen, this model proposed by Ceria and Stubbs [4] only considers uncertainty region on the mean vector of returns assuming the covariance matrix of returns is perfectly known. In the first part of the numerical results section, the effect of the estimated errors in expected returns is shown by efficient frontiers, and the improvement provided by using the robust optimization (4) is also presented.

2 Robust minimum variance

Recently, SDP has been used to compute Chebyshev-type upper and lower bounds firstly by Bertsimas and Popescu [1], then by Lasserre [7] and Vandenberghe et al. [14]. In which, the problem of computing tight bounds of a probability measure on a given semialgebraic set is converted into SDP problems and solved efficiently. Inspired by this work, we introduce the SDP formulations for the robust portfolio selection problem. Let us recall that the minimum variance formulation of the portfolio selection problem is usually defined as follows:

$$\begin{aligned} \min_{w \in \mathbb{R}^n} \quad & w^T \Gamma w \\ \text{s.t} \quad & w \in \bar{W}, \\ & w^T \hat{x} \geq R, \end{aligned}$$

where R is the lower limit on the expected return one would like to achieve. The robust portfolio selection problem as in Tütüncü and Koenig [13] then becomes a "min-max" problem:

$$
\begin{aligned}
\min_{w\in\mathbb{R}^n} \max_{\Sigma\in\Sigma_U,\hat{x}\in\hat{x}_U} \quad & w^T\Gamma w = w^T(\Sigma - \hat{x}\hat{x}^T)w \\
\text{s.t} \quad & w \in \bar{W}, \\
& \min_{\hat{x}\in\hat{x}_U} w^T\hat{x} \geq R,
\end{aligned}
\tag{5}
$$

where the covariance matrix is written as $\Gamma = \Sigma - \hat{x}\hat{x}^T$, Σ is the second moment matrix of returns. Σ_U and $\hat{x}_U$ are uncertain regions of Σ and $\hat{x}$, respectively. The notations will be detailed in section 2.1. In the following we first assume that the expected returns of the assets are known and fixed exactly, the SDP model is established to calculate the worst-case variance and solve the robust portfolio selection problem. Following in the same vein, we introduce componentwise bounded uncertain regions for the mean vector and for the second moment matrix. An SDP formulation and an SOCP-SDP semi-infinite formulation are proposed to solve the corresponding robust portfolio selection problem.

2.1 Notations and conventions

We are mainly using SDP to establish the model so that the variables under consideration are in the form of positive semidefinite symmetric matrices. In this section, we introduce our notations and describe the second moment and the localizing matrices of returns. It is important to introduce these matrices, because that only under appropriate conditions on the moment and localizing matrices the worst-case variance problem can be cast as an SDP problem. We denote $\mathcal{B}$ be the usual Borel σ-field of $\mathbb{R}^n$, $x \in \mathbb{R}^n$ represents the returns of individual assets in one time period$\frac{x_{t+1}-x_t}{x_t}$, $A \succeq 0$ if a symmetric matrix A is positive semidefinite, and use

$$S : \{x \in \mathbb{R}^n | \theta(x) = x^T P_i x + 2q_i x + r_i \geq 0, \quad i = 1, ..., m\}$$

to denote a compact semi-algebraic set $S \in \mathcal{B}$ under consideration, which will be the uncertainty sets over the second moment matrix of returns, where $\theta(x) = x^T P_i x + 2q_i x + r_i : \mathbb{R}^{2n} \to \mathbb{R}$ is a quadratic polynomial. Denote μ as a probability measure and let

$$\int_S x d\mu = \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \end{bmatrix}$$

and $\int_S xx^T d\mu = \Sigma$ be the first moment (mean) and the second moment of returns, respectively. The covariance matrix is denoted by

$$\Gamma = \Sigma - \hat{x}\hat{x}^T \succeq 0.$$

We have the second moment matrix denoted by

$$\hat{\Sigma} = \int_S \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}^T d\mu = \begin{bmatrix} \Sigma & \hat{x} \\ \hat{x}^T & 1 \end{bmatrix} \succeq 0,$$

We assume that $\hat{\Sigma} \succ 0$ and also $\Gamma = \Sigma - \hat{x}\hat{x}^T \succ 0$. We now introduce so-called localizing matrices (see e.g. Lasserre [7]) to express the set S in terms of the moment matrix. As the set S in this paper is described by quadratic polynomials $\theta(x)$, the localizing matrices are defined by

$$\begin{aligned} M_0(\theta y) &= \int_S \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} d\mu \\ &= \langle \hat{\Sigma}, \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \rangle \\ &= \langle \Sigma, P_i \rangle + 2q_i^T \hat{x} + r_i \geq 0, i = 1, \dots, m, \end{aligned}$$

where $\langle A, B \rangle = \mathbf{Tr}(AB)$ denotes the standard scalar product in the symmetric matrices, and $M_0(\theta y)$, a notation borrowed from Lasserre [7], represents the localizing matrices when the set S is described by quadratic polynomials $\theta(x)$. Note that by a result in Putinar [9] the positive semi-definiteness of the moment and localizing matrices are necessary and sufficient conditions for the elements of $\hat{\Sigma}$ to be the moments of some measure μ supported on S when the set S is compact. This result will be directly applied in this paper and we refer interested readers to Putinar [9] and Lasserre [7]. In addition, we introduce a mean square matrix denoted by $\bar{X} = \begin{bmatrix} \hat{x}\hat{x}^T & \hat{x} \\ \hat{x}^T & 1 \end{bmatrix}$, and a portfolio matrix denoted by $\hat{W} = \begin{bmatrix} ww^T & w \\ w^T & 1 \end{bmatrix}$,where $w \in W$ is the weights of a portfolio. Generally speaking, defining linear matrix variable $\begin{bmatrix} X & x \\ x & 1 \end{bmatrix} \succeq 0$ with variables X and x means $X - xx^T \succeq 0$ by Schur complement so that X is not necessary equal xx^T. However, according to an interesting result given by Body and Vandenberghe [2], we observe that quadratic programming problems defined by

$$
\begin{aligned}
\min_{x} \quad & \langle A_0, X \rangle + 2b_0^T x + c_0 \\
\text{s.t} \quad & \langle A_1, X \rangle + 2b_1^T x + c_1 \leq 0 \\
& X = xx^T,
\end{aligned}
$$

can be cast as an SDP problem defined by

$$
\begin{aligned}
\min_{X,x} \quad & \langle A_0, X \rangle + 2b_0^T x + c_0 \\
\text{s.t} \quad & \langle A_1, X \rangle + 2b_1^T x + c_1 \leq 0 \\
& \begin{bmatrix} X & x \\ x & 1 \end{bmatrix} \succeq 0,
\end{aligned}
$$

and where both $A_i, \ \ i = 0, 1$ are symmetric matrices, $b_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}$. These two problems have the same dual problem

$$\begin{aligned} \max_{\gamma,\lambda} \quad & \gamma \\ \text{s.t} \quad & \lambda \geq 0 \\ & \begin{bmatrix} A_0 + \lambda A_1 & b_0 + \lambda b_1 \\ (b_0 + \lambda b_1)^T & c_0 + \lambda c_1 - \gamma \end{bmatrix} \succeq 0. \end{aligned}$$

and the strong duality holds provided Slater's constraint qualification is satisfied, i.e. there exists an x with $\langle A_1, X\rangle + 2b_1^T x + c_1 < 0$. Note that this result is valid when matrices A_i are positive semidefinite or negative semidefinite. The proof employs so-called S-procedure and is elaborated in Body and Vandenberghe [2]. In this paper we will only encounter the convex case, which means $A_i \succeq 0$. Therefore, when we have quadratic programming (QP) problems as just mentioned, the linear matrix variable $\begin{bmatrix} X & x \\ x & 1 \end{bmatrix}$ in the SDP formulation is essentially equivalent to the quadratic matrix variable $\begin{bmatrix} xx^T & x \\ x & 1 \end{bmatrix}$ in the QP formulation. We will use the quadratic matrix variables $\hat{W}$ and $\bar{X}$ directly in the SDP formulation as they are more intuitive than their corresponding linear matrix variables in formulating the robust portfolio optimization problem.

2.2 Optimizing the second moment

In this section, suppose that the expected returns $\int_S x d\mu = \hat{x}$ are known and fixed, we propose a framework to cast the portfolio optimization problem with worst-case variance as an SDP problem. The covariance matrix of returns is $\Sigma - \hat{x}\hat{x}^T$, given the mean of returns $\hat{x}$, the worst-case variance is only dependent on the second moment Σ. Suppose that the portfolio w is fixed, then the worst-case problem can be formulated as follows:

$$\begin{aligned} \sup_{\mu} \quad & \langle ww^T, \int_S xx^T d\mu \rangle - \langle w, \int_S x d\mu \rangle^2 && (6) \\ \text{s.t} \quad & \int_S x d\mu = \hat{x}, \\ & \int_S d\mu = 1, \\ & \mu(x) \geq 0, \end{aligned}$$

where μ is a probability measure.

Theorem 2.1. *Suppose that S is compact, convex semialgebraic set with nonempty interior, then problem (6) is equivalent to the following problem:*

$$\begin{aligned} \max_{\Sigma} \quad & w^T \Gamma w = \max_{\hat{\Sigma}} \langle \hat{W}, \hat{\Sigma} \rangle - \langle \hat{W}, \bar{X} \rangle && (7) \\ \text{s.t} \quad & \hat{\Sigma} \succeq 0, && (7a) \\ & \langle A_i, \hat{\Sigma} \rangle = \hat{x}_i, \quad i = 1, ..., n+1, \\ & \langle B_j, \hat{\Sigma} \rangle \geq 0, \quad j = 1, ..., m. && (7b) \end{aligned}$$

The matrices A and B in this formulation are selected carefully to satisfy the constraints of the problem. For instance,

$$A_1 = \begin{bmatrix} 0 & ... & 0.5 \\ \vdots & \vdots & \vdots \\ 0.5 & ... & 0 \end{bmatrix}$$

specify $\hat{x}_1$ in the moment matrix $\hat{\Sigma}$. In the same way, suppose that the uncertain region S is described by $x^T x \leq 1$ so that matrix B is designed by $\begin{bmatrix} -\mathbf{I} & 0 \\ 0 & 1 \end{bmatrix}$, where $\mathbf{I}$ is the n-dimensional identity matrix. The problem above will calculate the worst-case variance given a portfolio w.

Proof. Note that the constraints (7a) and (7b) are showing positive semidefiniteness of the moment and localizing matrices and S is compact which, as aforementioned, guarantee that the elements of $\hat{\Sigma}$ are the moments of some measure μ supported on S. The main proof of this theorem follows directly from Theorem 3.7 in Lasserre [7].

□

The robust portfolio selection problem (5) therefore becomes a "min-max" problem and can be formulated as follows:

$$\begin{aligned} \min_{\hat{W}} \max_{\hat{\Sigma}} \quad & \langle \hat{W}, \hat{\Sigma} \rangle - \langle \hat{W}, \bar{X} \rangle \\ \text{s.t} \quad & \hat{\Sigma} \succeq 0 \\ & \langle A_i, \hat{\Sigma} \rangle = \hat{x}_i, \quad i = 1, ...n+1, \\ & \langle B_j, \hat{\Sigma} \rangle \geq 0, \quad j = 1, ..., m, \\ & \langle C, \hat{W} \rangle \in \bar{W}, \\ & \hat{W} \succeq 0 \\ & \langle D, \hat{W} \rangle \geq R, \end{aligned} \tag{8}$$

where C are designed matrices to fit in the set of feasible portfolios. For example, $C_1 = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$ and $C_2 = \begin{bmatrix} -\mathbf{I} & 0 \\ 0 & 1 \end{bmatrix}$ to satisfy $\bar{W} = \{w \in \mathbb{R}^n \,|\, \sum_{i=1}^n w_i = 1, w \geq 0\}$ and $w^T w \leq 1$. Note that the problem (8) is in fact a convex quadratic programming problem with respect to the portfolio variable w for every positive semidefinite matrices $\hat{\Sigma} - \bar{X}$ constrained by a quadratic function $w^T w \leq 1$. As demonstrated in section 2.1, the quadratic matrix variable $\hat{W}$ is equivalent to the linear matrix variable $\begin{bmatrix} W & w \\ w & 1 \end{bmatrix} \succeq 0$. The last constraint is the performance constraint in which $D = \begin{bmatrix} 0 & \frac{1}{2}\hat{x} \\ \frac{1}{2}\hat{x} & 0 \end{bmatrix}$ and R is the required return of the portfolio.

Lemma 2.1. *Suppose that strong duality holds for the max part of (8), the "min-max" problem (8) can then be expressed by the following "min" SDP problem :*

$$\begin{aligned}
\min_{y,\hat{W}} \quad & -\hat{x}^T y - \langle \hat{W}, \bar{X} \rangle \\
\text{s.t} \quad & -A^T y - B^T \tau - \hat{W} = \Lambda \qquad (9) \\
& \Lambda \succeq 0 \\
& \tau \geq 0, \quad \tau \in \mathbb{R}^m, \\
& \langle C, \hat{W} \rangle \in \bar{W}, \\
& \langle D, \hat{W} \rangle \geq R.
\end{aligned}$$

Proof. Proof of this lemma is mainly based on writing the Lagrangian of "max" function of (8) to achieve the dual (see e.g. Boyd and Vanderberghe [2]). □

By this lemma, we can solve the robust portfolio selection problem (5) by a single "min" SDP problem instead of a "min-max" problem (8). Without the last performance constraint, (9) gives the optimal portfolio assuming the worst-case variance of returns. If one wants to find out the worst-case mean and variance corresponding this optimal portfolio, the "max" part of (8), or in other words (7) needs to be solved by giving the portfolio w. By employing this framework, we now consider uncertain regions over both the expected returns and the second moment matrix of returns and solve the robust portfolio selection problem.

2.3 Optimizing the second moment and the mean

In this section, we relax the first moment constraints by considering componentwise bounds :$\hat{x} \in [\hat{x}_l, \hat{x}_u] \in \mathbb{R}^n$, where $\hat{x}_u, \hat{x}_l$ are given componentwise upper and lower bounds of the mean vector, respectively. By using a similar approach to that of section 2.2, we provide two equivalent robust formulations. One is based on "min-max" and solved by using semi-infinite programming. The other one is based on reformulation by using duality to an overall minimization problem.

Following the same procedures as presented previously, we start with the "max" function by assuming that portfolios w is given. The worst-case variance problem can be formulated as follows:

$$\max_{\Sigma,\hat{x}} \; w^T \Sigma w - w^T \hat{x}\hat{x}^T w = \max_{\hat{\Sigma},\bar{X}} \langle \hat{W}, \hat{\Sigma} \rangle - \langle \hat{W}, \bar{X} \rangle \tag{11}$$

$$\begin{aligned}
\text{s.t} \langle A, \mathbf{1}\hat{\Sigma} \rangle &\in [\hat{x}_l, \hat{x}_u] \,, \\
\langle A, \mathbf{1}\bar{X} \rangle &\in [\hat{x}_l, \hat{x}_u] \,, \\
\langle A, (\hat{\Sigma} - \bar{X}) \rangle &= 0, \\
\langle B_j, \hat{\Sigma} \rangle &\geq 0, \quad j = 1, ..., m, \\
\langle C, \hat{\Sigma} \rangle &= 1, \\
\langle C, \bar{X} \rangle &= 1, \\
\hat{\Sigma} &\succeq 0, \\
\bar{X} &\succeq 0, \\
\hat{\Sigma} - \bar{X} &\succeq 0,
\end{aligned}$$

where the matrices $A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$ again are designed to identify the mean vector $\hat{x}$ which belongs to $[\hat{x}_l, \hat{x}_u]$, and the matrix $C = \begin{bmatrix} 0 & ... & 0 \\ \vdots & \vdots & \vdots \\ 0 & ... & 1 \end{bmatrix}$ is to spot "1" in the matrices $\hat{\Sigma}$ and $\bar{X}$. To guarantee $\hat{\Sigma}$ to be the second moment matrix of some measure supported on S, we again have the positive semidefiniteness constraints of the moment matrix $\hat{\Sigma}$ and the localizing matrices $\langle B_j, \hat{\Sigma} \rangle$, $j =$

$1, ..., m,$. The constraint $\langle A, (\hat{\Sigma} - \bar{X}) \rangle = 0$ ensures that the mean vectors in the second moment matrix and in the mean square matrix are the same. Moreover, in this worst-case variance formulation we have a convex QP problem with respect to $\hat{x}$ constrained by $\hat{\Sigma} - \bar{X} \succeq 0$, which is again the situation we illustrated in section 2.1 where the quadratic matrix variable $\bar{X}$ is equivalent to the the linear matrix variable $\begin{bmatrix} X & x \\ x & 1 \end{bmatrix} \succeq 0$. As can be seen, through this formulation one can derive the worst-case variance of returns given reasonable componentwise bounds on the mean of returns. The robust portfolio selection portfolio problem therefore becomes a "min-max" problem:

$$\min_{w} \max_{\hat{\Sigma}, \bar{X}} \quad q(w, \bar{X}, \hat{\Sigma}) = \langle \hat{W}, \hat{\Sigma} - \bar{X} \rangle \tag{12a}$$

with constraints. In the sequel, we use two different approaches to solve the "min-max" problem. Firstly, following the exactly procedures introduced in the previous subsection, we convert the "min-max" problem to a "min" convex SDP problem. The second approach is by using semi-infinite programming (Zakovic and Rustem [11]). We use the second approach as an alternative to verify the validity of the first approach.

As to the first approach, we can, by using the similar lemma illustrated in section 2.2, express the problem (12a) in the form of the following:

$$\begin{aligned} \min_{\alpha_1, \alpha_2, \alpha_3, \tau, y_u, y_l, z_u, z_l, \Lambda, \hat{W}} \quad & \alpha_1 + \alpha_2 - \hat{x}_l^T (y_l + z_l) + \hat{x}_u^T (y_u + z_u) \\ \text{s.t} \quad & \alpha_1 C + \alpha_3 A - \tau B - y_l A + y_u A - \Lambda - \hat{W} = \Lambda_1, \\ & \alpha_2 C - \alpha_3 A - z_l A + z_u A + \Lambda + \hat{W} = \Lambda_2, \\ & \langle D, \hat{W} \rangle \in \bar{W}, \\ & \tau \geq 0, \\ & y_u, y_l, z_u, z_l \geq 0, \\ & \Lambda, \Lambda_1, \Lambda_2 \succeq 0, \\ & \langle E, \hat{W} \rangle \geq R, \end{aligned} \tag{12}$$

where matrices A,B and C are the same as illustrated previously, D is to ensure the feasible set of portfolios, and E is to suffice the performance constraint by the required return of the portfolio R.

Proof. The proof is again utilizing Langrangian to calculate its dual (see e.g. Boyd and Vanderberghe [2]). If the covariance matrix is positive definite so that strong duality holds according to the slater condition (see e.g. Boyd and Vanderberghe [2]). □

We can therefore, without the performance constraint in (12), attain the optimal portfolio assuming the worst-case mean and variance of returns by solving the single convex "min" SDP problem (12). As to the second approach, the "min-max" problem can also be written as

$$\min_{w} \max_{\hat{\Sigma},\bar{X}} \quad q(w,\bar{X},\hat{\Sigma}) = \begin{bmatrix} w^T & 0 \end{bmatrix} (\hat{\Sigma} - \bar{X}) \begin{bmatrix} w \\ 0 \end{bmatrix},$$

which means it is a convex quadratic programming problem with respect to $\begin{bmatrix} w \\ 0 \end{bmatrix}$ and a linear SDP problem with respect to $\hat{\Sigma}, \bar{X}$. By employing the semi-infinite programming algorithm introduced by S.Zakovic and B.Rustem [11], we can achieve the optimal portfolio and its corresponding worst-case mean and variance at the same time.

Semi-infinite Programming Algorithm:

Set $A = \{(\hat{\Sigma}_0, \bar{X}_0)\}$

while $U > L$ **do**

Compute lower bound $L = \min_{w \in W} \max_{\hat{\Sigma},\bar{X}} q(w,\bar{X},\hat{\Sigma})$ globally

with $w^* = \arg\min_{w \in W} \max_{(\hat{\Sigma},\bar{X}) \in A} q(w,\bar{X},\hat{\Sigma})$

Compute upper bound $U = \max_{(\hat{\Sigma},\bar{X}) \in B} q(w^*,\bar{X},\hat{\Sigma})$

with $(\hat{\Sigma}^*, \bar{X}^*) = \arg\max_{(\hat{\Sigma},\bar{X}) \in B} q(w^*,\bar{X},\hat{\Sigma})$

$A = A \cup \{(\hat{\Sigma}^*, \bar{X}^*)\}$

end while

Optimal values $(w^*, \hat{\Sigma}^*, \bar{X}^*)$ are attained when $U == L$

Note that the first "min-max" problem in the algorithm involving a maximum of norms can be cast as a standard SOCP problem: the problem

$$\min_{\hat{w}} \max_{\hat{\Gamma} \in \bigcup_{i=0,\dots,p} (\hat{\Sigma}_i, \bar{X}_i)} \hat{w}^T \hat{\Gamma}_i \hat{w} = \left\| \hat{\Gamma}_i^{1/2} \hat{w} \right\|^2$$

$$\text{s.t} \quad w \in \bar{W},$$

which is related to the SOCP problem

$$\begin{aligned} \min \quad & t \\ \text{s.t} \quad & \left\| \hat{\Gamma}_i^{1/2} \hat{w} \right\| \leq t, i = 0, ..., p \\ & w \in \bar{W}, \end{aligned}$$

where variables $w \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\hat{w}$ is designed to be $\begin{bmatrix} w \\ 0 \end{bmatrix}$ to suit the matrices $\hat{\Sigma}$ and $\bar{X}$ for calculations. In each iteration, the lower bound $L = t^2$ is computed to compare with the upper bound U generated by the second maximization problem. The second maximization problem in the algorithm is a linear SDP problem with domain B that can be solved efficiently. After each iteration, if $U > L$ then the set A is enlarged by the new $(\hat{\Sigma}^*, \bar{X}^*)$ generated by the second maximization problem, which will cause p increases one in the SOCP problem in the next iteration. The iterations terminate when $U = L$. Our numerical experiments show that the almost same results (accuracy denoted by $1E-4$) can be attained by solving the robust portfolio selection problem via these two approaches aforementioned. It can be seen that the single SDP approach is easier and faster to solve the robust portfolio selection problem. However, one needs to plug in the optimal portfolio into

(11) to attain the corresponding worst-case variance (the worst-case mean and second moment). the semi-infinite formulation is an alternative way to solve the robust portfolio selection problem and also provides the validity of the SDP model. To this end, we can use the following algorithm to generate a discrete approximation to the robust efficient frontier:

Robust Efficient Frontier Algorithm:

1. Solve problem (9) or (12) without performance constraint to attain optimal portfolio w_{min}.
2. Solve problem (7) or (11) to attain the worst-case mean and variance $\hat{x}_{wc}, \Gamma_{wc}$.
3. Set $R_{min} = \hat{x}_{wc}^T w_{min}$, $R_{max} = \hat{x}_{wc}^T w_{max}$ and $\Delta = R_{max} - R_{min}$.
4. Choose N, the number of desired points on the efficient frontier. For $R \in \{R_{min} + \frac{\Delta}{N-1}, R_{min} + 2\frac{\Delta}{N-1}, ..., R_{min} + (N-1)\frac{\Delta}{N-1}\}$ solve problem (12) with constraints $\hat{x}_{wc}^T w = R$.

In this algorithm, w_{min} corresponds to the risk averse portfolio with respect to the worst-case risk measure $\min_w \max_{\hat{\Sigma}, \bar{X}} \ q(w, \bar{X}, \hat{\Sigma})$ without performance requirement of the type of the last constraint in (12) on the portfolio. This essentially corresponds to the worst-case portfolio performance. w_{max}, on the other hand, represents to the best portfolio return with respect to the worst-case $\hat{x}$ with no consideration on risk. This is in fact the problem $\max_w \min_{\hat{x}} \ \hat{x}^T w$ with constraints only on $\hat{x}$ and w.

3 Numerical Results

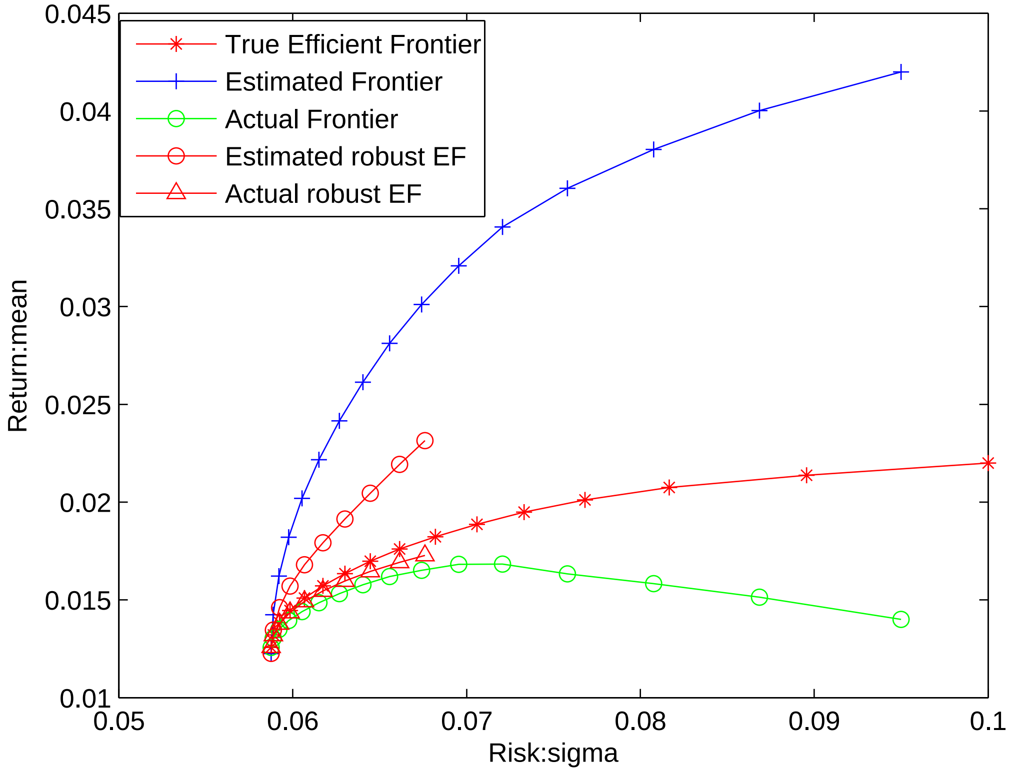
We now illustrate numerical results of the proposed methods to the problems of robust portfolio optimization. The SOCP and SDP solver used is SeDuMi version 1.1 developed by Sturm ([10]). The computations are done on a pentium IV 3.2G HZ PC with 1G RAM.

The first numerical experiment is to investigate the effect of the errors of the estimated expected returns (see Ceria and Stubbs [4]). We take Broadie [3]'s true and estimated data as inputs and generate the estimated, true and actual efficient frontiers. We then employ (4) with the estimated expected returns and true expected returns to calculate the estimated robust efficient frontier and the actual robust efficient frontier. Suppose that the true expected returns falls in the confidence region with probability 95%, which means $k = 1.0703$ in the formulation (4), we have the efficient frontiers shown in the Figure 1. We can see that the robust portforlios perform better than MMV's portfolios in terms of sensitivity to the input data. The numerical results also show that the robust portfolios are more diversified than MMV's portfolios.

The following numerical experiments show that the effect of robust portfolios over uncertainty of the covariance matrix of returns, and the robust efficient frontiers are presented. When the expected returns are specified, we optimize the portfolios assuming the worst-case variance, which is, in fact, an implementation of the proposed model (9). For simplicity, suppose that $S := \{x \in \mathbb{R}^5 | x^T x \leq 0.1\}$ and the expected returns and the covariance matrix of returns are taken from Broadie [3],we can compute an efficient frontier of the MMV model and the worst-case variances with given portfolios via (7). Moreover, by employing the robust efficient frontier algorithm and the formulation (9) the optimal portfolios are achieved assuming the worst-case variances. The result is shown in Figure 2.

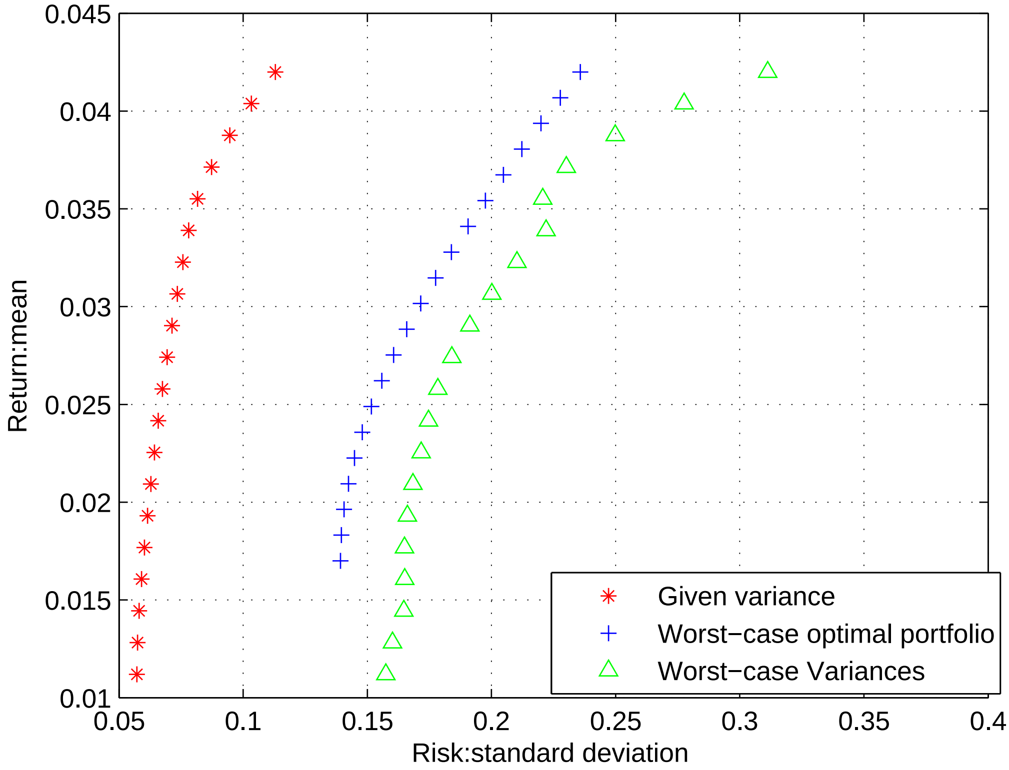
When the expected returns are only given by componentwise upper and lower bounds, we implement the two models proposed previously to solve the robust portfolio selection problem. The approach by employing semi-infinite programming algorithm can attain the worst-case expected returns, the covariance matrix of returns and the corresponding optimal portfolio simultaneously, but it is slower then using the linear SDP model ((11) and

Figure 1: Efficient Frontiers



(12)). By choosing upper bounds and lower bounds of the expected returns, the semi-infinite algorithm needs about 90 iterations to attain the optimal value. On the other hand, we can solve an single convex SDP problem (12) to gain the optimal portfolio assuming the worst-case variance. To this end, the classical efficient frontier and the robust efficient frontier generated by the algorithm aforementioned are shown in the Figure 3. Note that the Figure 3 only demonstrates how different the robust efficient frontier may look comparing with the classical efficient frontier. The performance of the robust portfolio is largely dependent on the settings of the uncertainty regions. We will test our robust portfolio model in the following backtesting.

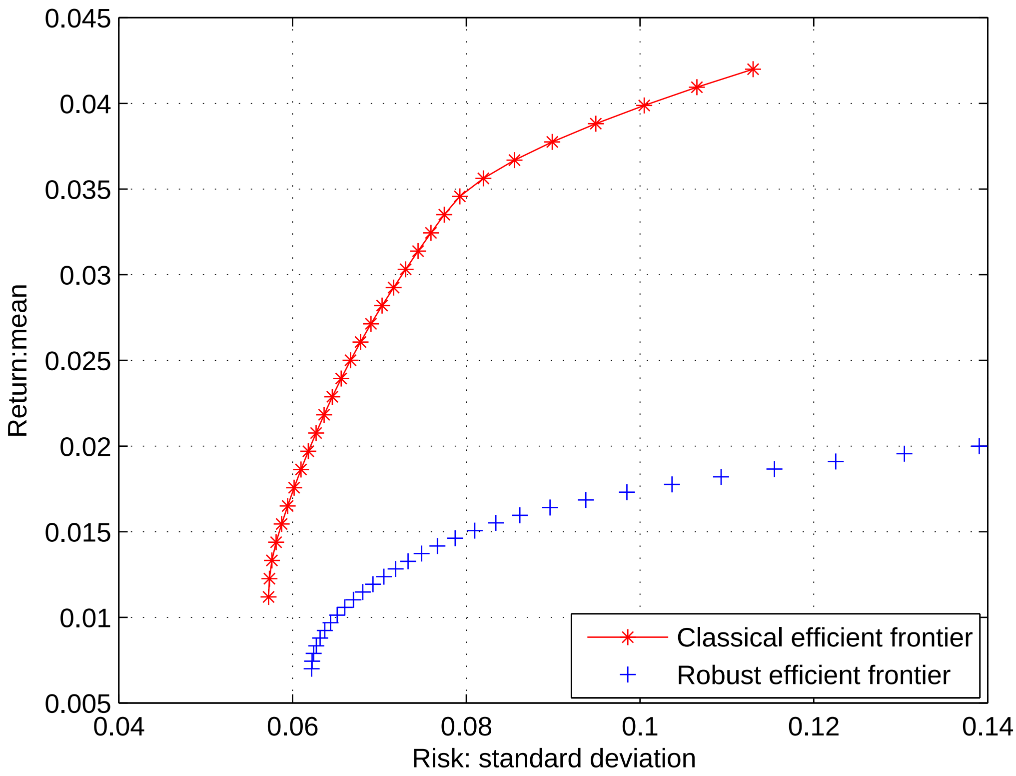
Figure 2: Efficient Frontier, worst-case variance, and worst-case optimal portfolio



3.1 Backtesting

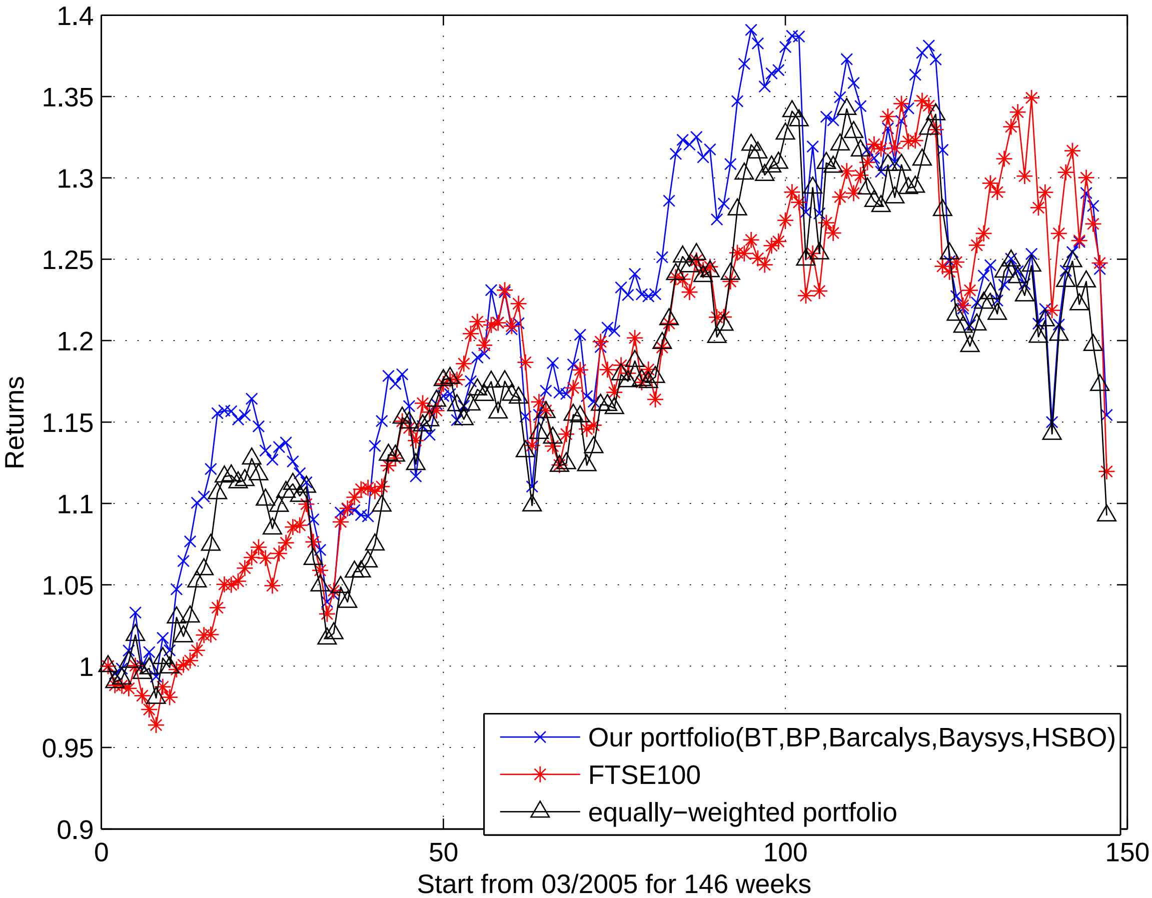
We randomly choose 5 components (BT, BP, Barcalys, Bay system, HSBO) of FTSE100 as our portfolio and employ the robust portfolio optimization and selection model proposed in this paper to compare the returns with the index FTSE100 from 3rd January 2005 to 22th January 2008. We take 10 weeks as a history window from which we calculate means, covariance matrices, and the second moment matrices every week (5 days). With the highest and the lowest value over these weeks as the elements of the upper or lower bound of mean vector and the second moment matrix, we can obtain, according to (12), a robust portfolio which assumes that in the future week

Figure 3: Robust Efficient Frontier



the mean and the second moment matrix of returns would not escape from the bounds provided. The history windows keeps moving which means that we are always looking back for the latest 10 weeks in order to construct a portfolio for the next week. In running the backtesting, we occasionally encounter numerical problems with the solver SEDUMI (see e.g. Sturm [10]), we then switch to SDPT3 (see e.g. Toh et al. [12]) to solve the problem. Equally weighted portfolio is also included in the backtesting as it can be obtained when S is described by a sphere. The result is shown in the Figure 4.

Figure 4: Backtesting



4 Conclusion

We have established the one time period robust portfolio selection model under the setting of conic programming. Uncertain regions are introduced to both the expected returns and the second moment matrix of returns, the resulting robust portfolio selection problems are formulated as conic programming problems and solved by solvers available in public domain. It is shown that the robust portfolios perform more reliable than MMV's portfolios in the sense that the robust portfolios are less sensitive to the errors of inputs.

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