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Abstract

Recently, given the first few moments, tight upper and lower bounds of the no arbitrage prices can be obtained by solving semidefinite programming (SDP) or linear programming (LP) problems. In this paper, we compare SDP and LP formulations of the European-style options pricing problem and prefer SDP formulations due to the simplicity of moments constraints. We propose to employ the technique of change of numeraire when using SDP to bound the European type of options. In fact, this problem can then be cast as a truncated Hausdorff moment problem which has necessary and sufficient moment conditions expressed by positive semidefinite moment and localizing matrices. With four moments information we show stable numerical results for bounding European call options and exchange options. Moreover, A hedging strategy is also identified by the dual formulation.

Keywords: moments of measures, semidefinite programming, linear programming, options pricing, change of numeraire

1 Introduction

Pricing financial derivatives has been a major focus in financial engineering. One of the central questions in this area is, given information on the underlying assets, to find no arbitrage prices for derivatives of such underlying assets. Black-Scholes formula provides an insightful answer to this question. However, the result is based on the assumption that the underlying price dynamics follows a geometric Brownian motion, which often becomes a target of critics. Corresponding to this, instead of pursuing an exact value, under assumptions of no-arbitrage and a complete market, we try to achieve the tight bounds of the no-arbitrage option prices by using the moment approach.

Several authors have tried to use tools of semidefinite programming(SDP) and linear programming(LP) to obtain bounds on option prices. In particular, Boyle and Lin [12] is, to our knowledge, the first to use SDP to bound the option on maximum of multiple assets with first two moments information. Bertsimas and Popescu [3] are among the first to solve the moment problems using SDP systematically and apply this framework to compute the tight bounds of European options (one dimension). Lasserre [7] then establishes a scheme of semidefinite approximations for multi-dimensional cases and later applies it in a class of exotic options (see Lasserre et al. [16]). The main issue in the moment problems with SDP is to define constraints that guarantee a nonnegative polynomial on $\mathbb{R}^n$ can be decomposed into a sum of squares of other polynomials, which is Hilbert's 17th problem presented at Paris in 1900. Bochnak et al[10] give a description of the problem in special cases: $n \leq 2; m = 2$; or $n = 3, m = 4$; where n and m denote number of dimensions and moments, respectively. In the general case, Renznick [13] shows that if $\theta(x) \in \mathbb{R}^n$ is positive definite it is always possible to write $(\sum_{i=1}^n x_i^2)^r \theta(x)$ for a certain $r \in \mathbb{N}$ as a sum of squares. A hierarchy of semidefinite approximations for $P_{n,m}$ is therefore derived by Zuluaga et al [17]. On the other hand, the semidefinite approximation scheme developed by Lasserre [7] is exploiting Putinar's theorem [11] which assumes a semialgebraic compact set defined by polynomials. In this paper, we will employ Putinar's theorem to bound option prices due to its simplicity of implementation. We present Putinar's theorem and the SDP formulations in section 2.1. With regard to the approach of LP, Stockbridge [14] proposes infinite-dimensional LP formulations for different options, which can lead us to SDP or LP problems. We will find out in section 2.2 that despite the advantages of LP solvers, we prefer to use SDP to bound option prices due to its simplicity of moment constraints.

In this paper, we propose to apply the technique of change of numeraire on bounding equity option prices with SDP in one and two dimension cases. The change of numeraire technique was first introduced by Jamshidian [6] to deal with interest rate derivatives. In the case of equity derivatives, the numeraire is usually taken by the cash bond, but it can be any of the tradable instruments. It is also well known that no matter which numeraire is chosen, the price of the derivative will always be the same. The idea of changing numeraire comes after observing unstable numerical results by direct implementing the SDP models aforementioned, especially when higher moments (for e.g. up to 4th moments) information are involved. According to our numerical experiments, one often has a marginally feasible problem when dealing the truncated Stieltjes moment problem (unbounded region) with higher moments, which may cause the instability of the numerical results. With the change of numeraire we are able to cast the bounding option price problem as the truncated Hausdorff moment problem (bounded region) rather than the truncated Stieltjes moment problem (unbounded region) (see e.g. [16]), which shows much more stable numerical performance. Another reason to employ the change of numeraire technique is due to the interesting analysis of bounds on measures with support region on a compact interval in the one dimensional case by Lasserre [7]. The work shows that only under at least four moment conditions a tight upper bound will be more discriminating. We will show in the numerical experiments that the bounds computed via SDP are indeed tighter with first four moments than with only first two moments. We also show that change of numeraire can also simplify options pricing problems, for instance, the exchange option by reducing the dimensions of the problem comparing with the work by Zuluaga and Peña [18]. Moreover, we identify a hedging strategy from the SDP formulations and apply our methods to two types of options including European call options and exchange options in section 3. Our numerical results in section 3 are numerically computed to be global optimal using Gloptipoly3 (SeDuMi). In the following, we first study the SDP and LP formulations for the the European-style option prices problem in general and demonstrate that SDP is preferred as the moments constraints are easy to be expressed. As we want to obtain tighter bounds with higher moments conditions, we propose and apply the technique of change of numeraire to cast the bounding option prices problem as a Hausdorff moment problem and numerical results are presented. Notice that the moments conditions of the martingale measure (risk-neutral) should be collected and calculated from the existed option prices, but for the convenience of comparison with the Black and Scholes' closed-form solution and ease of computation, we calculate the moments from the exponential martingales.

2 Formulation

In this section, we consider European-style option pricing problem with knowing a few moments information in general (multi-dimension). In particular, we have an option with payoff function $f(x), f : \mathbb{R}^n \to \mathbb{R}$, and the tight bounds on the price of European-style option can be formulated as follows:

$$\begin{aligned} \min \text{ or } \max \;\; & \mathbb{E}_\mu[f(x)] \\ \text{s.t.} \;\; & \int x^\alpha d\mu = \sigma_\alpha, \quad \forall \alpha \in \mathbb{I}_d \\ & \int \mu(x) dx = 1, \\ & \mu(x) \geq 0, \quad x \in \mathbb{R}^n, \end{aligned} \tag{4.1}$$

where the expectation is taken over all martingale measures μ defined on $\mathbb{R}^n$ and σ_α are the truncated moments of the measure. $\mathbb{I}_d$ is a finite set defined by $\{\alpha \in \mathbb{N}^n : \alpha_1 + \alpha_2 + ... + \alpha_n \leq d\}$. In the sequel, we will introduce SDP and LP formulations to solve this kind of problems.

2.1 SDP: Primal and Dual

As stated by many aforementioned literatures (see e.g. [3]), the problem (4.1) can be formulated as follows:

$$\begin{aligned} & \inf_\mu \text{ or } \sup_\mu \int_S f(x) d\mu \\ (P) \qquad & \text{s.t.} \int_S x^\alpha d\mu = \sigma_\alpha, \quad \forall \alpha \in \mathbb{I}_d \\ & \mu(x) \in M(S), \end{aligned}$$

where $f(x) : S \to \mathbb{R}$, a real-valued measurable function, is mainly a piecewise linear polynomial on S. $\mathcal{B}$ is the Borel σ-field of $\mathbb{R}^n$, $S \in \mathcal{B}$ is the domain under consideration, and $M(S)$ denotes the set of finite positive Borel measures supported by S. Therefore, the problem can be seen as computing upper or lower bounds of expectations of functions $f(x)$ with moments constraints on x in the domain S. In the sequel, we will only demonstrate, for convenience of exposition, the case of upper bound (sup), the lower bound follows in an analogous manner. We hence can obtain the primal and dual as follows:

$$
\begin{array}{llll}
 & z_P = \sup_{\mu} \int_S f(x) d\mu & & z_D = \inf \sum_{\alpha \in \mathbb{I}_d} \sigma_\alpha \theta_\alpha \\
(P) & \text{s.t.} \int_S x^\alpha d\mu = \sigma_\alpha, \quad \forall \alpha \in \mathbb{I}_d & (D) & \text{s.t.} \sum_{\alpha \in \mathbb{I}_d} \theta_\alpha x^\alpha \geq f(x) \quad \forall x \in S. \\
 & \qquad \mu(x) \in M(S). & &
\end{array}
$$

Note that these formulations involve two well-studied classes of cones, namely, the moment cones and the cones of positive semidefinite polynomials. The cone of moments supported in S for the primal is defined as

$$M_{n,d}(S) = \{y \in \mathbb{R}^{\mathbb{I}_d} : y_\alpha = \mathbb{E}_\mu(x^\alpha), \forall \alpha \in \mathbb{I}_d \text{ for some } \mu \in M(S)\}.$$

The cone of positive semidefinite polynomial for the dual is defined as

$$P_{n,d}(S) = \{\theta = (\theta_\alpha)_{\alpha \in \mathbb{I}_d} \in \mathbb{R}^{\mathbb{I}_d} : \theta(x) = \sum_{\alpha \in \mathbb{I}_d} \theta_\alpha x^\alpha \geq 0 \ \ \forall x \in S\}.$$

Therefore, the sufficient conditions for the strong duality can be summarized as follows (see e.g. Zuluaga and Peña [18]):
If either

1. $\sigma \in \text{Int}(M_{n,d}(S))$ or

2. there exists $\theta \in \mathbb{R}^{\mathbb{I}_d}$ such that$(\theta - f) \in \text{Int}(P_{n,d}(S))$.

Then $z_P = z_D$. Here, $\text{Int}(S)$ denotes the interior of the set S. Moreover, if the function $f(x)$ is defined by a piece-wise n-variate polynomial of degree d, such as $f(x) : S = \cup_{i=1}^p S_i \to \mathbb{R}$, where Borel measurable sets $S_i \subseteq \mathbb{R}^n, \quad i = 1, ...p$, is defined by $f(x) = q^i(x)$ if $x \in S_i, i = 1, ..., p.$, the strong duality holds similarly with the previous case. The proof follows Zuluaga and Peña [18]. As mentioned in the introduction, by employing

Putina's theorem we have semidefinite representation results for both the moment cones (see e.g. [7]) and the cones of positive semidefinite polynomials (see e.g. [2]). Putinar's theorem is the cornerstone of the construction of Lasserre's hierarchy of semidefinite approximations, we state the theorem as follows.

Theorem 2.1. *Suppose that the semialgebraic set S defined by*

$$S := \{x \in \mathbb{R}^m | p_j(x) \geq 0, j = 1, ..., l\}$$

is compact, and there is a polynomial $p : \mathbb{R}^m \to \mathbb{R}$

$$p(x) = s_0(x) + \sum_{i=1}^{l} p_i(x) s_i(x), \forall x \in \mathbb{R}^m$$

such that the set

$$\{x \in \mathbb{R}^m | p(x) \geq 0\}$$

is compact and the polynomials $s_i(x), i = 0, ..., l$are all sums of squares. Then any polynomial $v(x)$, strictly positive on S, can be written as

$$v(x) = u_0(x) + \sum_{i=1}^{l} p_i(x) u_i(x), \forall x \in \mathbb{R}^m$$

for some polynomials $u_i(x), i = 1, ..., l$ that are all sums of squares.

Proof. The proof is referred to Putinar [11]. □

This theorem shows us that with the defined semi-algebraic compact set and mild conditions, we are able to decompose a positive polynomial into sum of squares of other polynomials. Based on this theorem, both the primal and dual problems mentioned above can be cast as SDPs and solved efficiently. Moreover, as we usually encounter piece-wise (linear) functions in option pricing problems, the underlying probability measure then are decomposed into several measures supported by different pieces. The following is the SDP formulations for such problems:

$$
\begin{array}{llll}
 & z_{P_{pw}} = \sup \sum_{i=1}^{p} \langle q^i, y^i \rangle & & \sup\langle c, x\rangle \\
(P_{pw}) & \text{s.t.} \sum_{i=1}^{p} y^i = \sigma, & (P_{SDP}) & \text{s.t.} Ax = b, \\
 & y^i \in \overline{M_{n,d}(S_i)}, \quad i = 1..., p. & & x \in \mathcal{K},
\end{array}
$$

$$
\begin{array}{llll}
 & z_{D_{pw}} = \inf\langle \sigma, \theta \rangle & & \inf\langle b, y\rangle \\
(D_{pw}) & \text{s.t.} \theta - q^i \in P_{n,d}(S_i), \quad i = 1, ...p, & (D_{SDP}) & \text{s.t.} A^* y - c \in \mathcal{K}^*,
\end{array}
$$

where $\overline{M}$ denotes the closure of M, $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map and $\langle Ax, y\rangle = \langle x, A^* y\rangle$, $\mathcal{K}$ and $\mathcal{K}^*$are a closed convex cone and its dual cone, respectively. (P_{SDP}) and (D_{SDP}) are standard SDP formulations in Sturm [15] when the cone $\mathcal{K}$ is defined by positive semidefinite symmetric matrices. It is worth emphasizing that the definition of the pieces of the support region of the measure μ is important in problems of bounding option prices as the objectives of such problems are often piece-wise linear functions. For instance, in the case of European call option (one dimension), we usually have two pieces in (P_{pw}) $p = 2$, and $q^1(x) = x - K, \quad q^2(x) = 0$ corresponding to support region $S_1 = [K, \infty), \quad S_2 = (\infty, K]$, respectively. In fact, in this case we are calculating the upper bound of the measure supported on $[K, \infty)$ given the the moments of the measure supported on $(-\infty, \infty)$. Note that y^i is the truncated moments of the measure on support region i, and hence the moment constraints become $y^1_\alpha + y^2_\alpha = \sigma_\alpha$, and $y^i \in \overline{M_{n,d}(S_i)}, i = 1, 2$. To solve the problem by SDP, the only remaining problem is to transfer constraints $y^i \in \overline{M_{n,d}(S_i)}, \quad i = 1..., p$, to linear matrices inequalities (LMIs) so it falls in with the standard SDP formulations. In other words, we need the constraints of positive semidefinite matrices to guarantee the sequence y^i is indeed the moments of some measure supported on S_i. This problem is usually referred as the truncated Hausdorff moment problem or the truncated Stieltjes moment problem depending on the supported region of the measure. If the supported region is compact, e.g. $[a, b]$ on the real line, it is termed by the truncated Hausdorff moment problem. If the support region is semi-compact, e.g. $[a, +\infty]$ on the real line, it is called the truncated Stieltjes moment problem. By a result of the aforementioned Putinar's theorem, the positive semidefinitness of appropriate moment and localizing matrices proved to be necessary and sufficient conditions for the sequence y to be the moments of some measure supported on a semi-algebraic compact set S (see e.g. [1], [7]). For easy and clear exposition, we use the similar notations as in Lasserre [7] and demonstrate the moment conditions in form of positive semidefinite matrices. Let

$$(x^\alpha, |\alpha| \leq 2r) := \left(1, x_1, ..., x_n, ..., x_1^2, x_1x_2, ..., x_1^d, x_1^{d-1}x_2, ..., x_n^{2d}\right), \quad (4.2)$$

be the basis of the space of real-valued polynomials in n variables, of degree at most $2d$, where $|\alpha| = \alpha_1 + ... + \alpha_n$. Given a multi-index family of scalars $\tilde{y} \equiv \{y_\alpha, \alpha \in \mathbb{N}^n\}$, let $\hat{y} \equiv \{\hat{y}_i, i \in \mathbb{N}\}$ denote the sequence obtained by ordering $\tilde{y}$ so that it conforms with the indexing implied by the basis(4.2). The moment-matrix $M_d(\tilde{y})$ with rows and columns indexed in the basis(4.2) is defined by

$$M_d(\tilde{y})(1, i) = M_d(\tilde{y})(i, 1) = \hat{y}_{i-1}, \quad for\ i = 1, ..., d + 1,$$

$$M_d(\tilde{y})(1, j) = y_\alpha\ and\ M_d(\tilde{y})(i, 1) = y_\beta \Rightarrow M_d(\tilde{y})(i, j) = y_{\alpha+\beta},$$

where $M_d(\tilde{y})(i,j)$ is the (i,j)-entry of the matrix $M_d(\tilde{y})$. To fix ideas, when $n=2, d=2$, one obtains $\hat{y} = \{y_{0,0}, y_{1,0}, y_{0,1}, y_{2,0}, y_{1,1}, y_{0,2} ...\}$ and

$$M_2(\tilde{y}) = \begin{bmatrix} y_{0,0} & | & y_{1,0} & y_{0,1} & | & y_{2,0} & y_{1,1} & y_{0,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{1,0} & | & y_{2,0} & y_{1,1} & | & y_{3,0} & y_{2,1} & y_{1,2} \\ y_{0,1} & | & y_{1,1} & y_{0,2} & | & y_{2,1} & y_{1,2} & y_{0,3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{2,0} & | & y_{3,0} & y_{2,1} & | & y_{4,0} & y_{3,1} & y_{2,2} \\ y_{1,1} & | & y_{2,1} & y_{1,2} & | & y_{3,1} & y_{2,2} & y_{1,3} \\ y_{0,2} & | & y_{1,2} & y_{0,3} & | & y_{2,2} & y_{1,3} & y_{0,4} \end{bmatrix}.$$

In this context, moment matrices are of relevance if the family of scalars $\tilde{y} \equiv \{y_\alpha, \alpha \in \mathbb{N}^n\}$ considered above can be identified with the moments of a finite measure μ defined on the Borel σ-algebra on $\mathbb{R}^n$. In such a case, given any $d \in \mathbb{N}$, the moment matrix $M_d(\tilde{y})$ is positive semidefinite, denoted $M_d(\tilde{y}) \succeq 0$ (similarly, the notation $\succ 0$ represents positive definite matrices). In fact, for all polynomials $x \to f(x)$ of degree at most k, and with vector of coefficients $(f_\alpha, |\alpha| \leq d)$ in the basis (4.2), we have

$$\langle f, M_d(\tilde{y}) f \rangle = \int f^2 d\mu \geq 0.$$

Note that the converse is not in general true: given a moment-like matrix $M_d(\tilde{y}) \succeq 0$, the y_α involved are not necessarily moments of some measure μ on $\mathbb{R}^n$.

We need also to introduce localizing matrices in order to take some bounded regions into consideration besides the general case of $\mathbb{R}^n$ we just mentioned above. Give a polynomial q, we consider the set $S \subseteq \mathbb{R}^n$ defined by

$$S = \{x \in \mathbb{R}^n | \ \ q(x) \geq 0\}.$$

The localizing matrix $M_d(q, \tilde{y})$ is defined as follows. Let $\beta(i,j)$ be the β-subscript of the (i,j)-entry of the matrix $M_d(\tilde{y})$. If the polynomial q has coefficients (q_α) in the basis 4.2, then the localizing matrix is defined by

$$M_d(q,\tilde{y})(i,j) = \sum_\alpha q_\alpha y_{\beta(i,j)+\alpha}.$$

For example, if $x \to q(x) : 1 - x_1^2 - x_2^2$, for $x \in \mathbb{R}^2$, then $M_1(q,\tilde{y})$ is

$$M_1(q,\tilde{y}) = \begin{bmatrix} 1 - y_{2,0} - y_{0,2} & y_{1,0} - y_{3,0} - y_{1,2} & y_{0,1} - y_{2,1} - y_{0,3} \\ y_{1,0} - y_{3,0} - y_{1,2} & y_{2,0} - y_{4,0} - y_{2,2} & y_{1,1} - y_{3,1} - y_{1,3} \\ y_{0,1} - y_{2,1} - y_{0,3} & y_{1,1} - y_{3,1} - y_{1,3} & y_{0,2} - y_{2,2} - y_{0,4} \end{bmatrix}.$$

Following the same argument, if the elements of the family $\tilde{y} \equiv y_\alpha$ are the moments of some measure μ supported on S, then $M_d(q,\tilde{y}) \succeq 0$, because, for all polynomials $x \to f(x)$ of degree at most d, and with vector of coefficients $(f_\alpha, |\alpha| \leq d)$ in the basis (4.2),

$$\langle f, M_d(q,\tilde{y})f \rangle = \int f^2 q d\mu \geq 0.$$

The converse is again not true: the necessary conditions $M_d(q,\tilde{y}) \succeq 0$ and $M_d(\tilde{y}) \succeq 0$ are not in general sufficient to ensure that the elements of $\tilde{y}$ are the moments of some measure μ supported on S. However, if S is a compact semi-algebraic set as we defined in the Putinar's theorem (Theorem 2.1):

$$S := \{x \in \mathbb{R}^n | g_i(x) \geq 0, forall i = 1, ..., l\},$$

where $g_i, \quad i = 1, ..., l$, are given polynomials, under some mild conditions, the conditions

$$M_d(\tilde{y}) \succeq 0 \quad \text{and} \quad M_d(g_i,\tilde{y}) \succeq 0, \quad i = 1, ..., l, \quad k = 1, 2, ..., \tag{4.3}$$

are necessary and sufficient for the elements of $\tilde{y}$ to be moments of some measure supported on S. Note that if S is not compact then the conditions (4.3) are only necessary but not sufficient. In particular, when on the real line ($n = 1$) we have following results stated by Lasserre et al. [16] and proved by Curto and Fialkow [1].

Theorem 2.2. *Given a vector* $y = (y_0, y_1, ..., y_{2d}) \in \mathbb{R}^{2d+1}$*, the following statements are true: (a) With regard to the truncated Hausdorff moment problem,*

$$M_d(y) \succeq 0 \quad and \quad M_d(g, y) \succeq 0,$$

with $x \to g(x) := (b - x)(x - a)$*, are necessary and sufficient conditions for the elements of* y *to be the first* $2d + 1$ *moments of a measure supported on* $[a, b]$*.*
(b) With regard to the truncated Stieltjes moment problem,

$$M_d(y) \succ 0 \quad and \quad M_d(g, y) \succ 0,$$

with $x \to g(x) := x - a$*, are sufficient conditions for the elements of* y *to be the first* $2d + 1$ *moments of a measure supported on* $[a, +\infty)$*.*

Note that this important result provides easy expressing conditions for measures supported on compact and non-compact sets, and we shall see later that this is one of the advantages of the SDP moment approach over linear programming (LP) approach in which one must consider moments of measure with supports on compact sets.

2.2 LP formulation

To solve the option prices problem, the basic infinite-dimensional LP formulation proposed by Stockbridge [14] is as follows:

$$\begin{aligned} \text{Optimize}_{\upsilon_T, \mu_0} \quad & \langle g(Z_T), \upsilon_T \rangle & (4.4) \\ \text{s.t} \quad & \langle f, \upsilon_T \rangle - \langle \mathcal{A} f(Z_t), \mu_0 \rangle = f(z), \forall f \in \mathcal{D} & \text{(bj)} \\ & |\upsilon_T| = 1, |\mu_0| = T, \end{aligned}$$

where the process satisfies

$$dZ_t = b(Z_t)dt + \sigma(Z_t)dB_t, \quad Z_0 = z_0 \in \mathbb{R}^n$$

in which $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are deterministic functions such that the diffusion has a unique strong solution. $\mathcal{A}$ is the infinitesimal generator of the underlying process Z_t that is defined by

$$f \to (\mathcal{A}f)(z) := \frac{1}{2}tr[\sigma\sigma^T f''](z) + [b^T f'](z), \quad f \in \mathcal{D}(\mathcal{A})),$$

the domain $\mathcal{D}(\mathcal{A}))$ of which contains the set $C^2(\mathbb{R}^n)$ of all twice-continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ with compact support. v_T denotes the distribution of Z_T called the exit location measure and μ_0 denotes the expected occupation measure of the diffusion Z up to time T. $|v_T|$ and $|\mu_0|$ denote the total masses of the measures v_T and μ_0, respectively. Moreover, the equation (bj) in (4.4) is called basic adjoint equation (e.g., see Helmes et al [5]) and "Optimize" stands for either "maximize" or "minimize". It can be seen that the basic adjoint equation in fact provides a way to calculate the moments of the exit location measure. If these moment are easy to be computed, this formulation coincides with the SDP formulation (primal) previously illustrated. Moreover, this formulation is more general in the sense that we can change the time to maturity T to $\mathcal{F}_t$-stopping time τ subject to constraints. Hence we can take path-dependent exotic options into account such as barrier options and Asian options (see e.g. [16]).
Notice that both formulations are originally infinite-dimensional LP formulations, with a finite number of moment constraints involved we are able to obtain finite relaxations of these formulations, one can end up with either SDP formulations(introduced in the previous section) or LP formulations. As to LP relaxations we have Hausdorff moment conditions which state that for any measure μ on $[0, 1]$

$$\int y^k(1-y)^n\mu(dy) \geq 0 \quad k = 0, 1, 2, \ldots; \quad n = 1, 2 \ldots.$$

Let $\{m_k\}_{k\geq 0}$ be the moments of μ. This inequality becomes:

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}m_{k+r} \geq 0 \quad k = 0, 1, 2, \ldots; \quad n = 1, 2 \ldots.$$

These moment conditions are necessary and sufficient conditions for the sequence $\{m_k\}_{k\geq 0}$ be the moments of μ, and are linear so they fit into the LP constraints. The advantage of LP is that there are many good LP solvers that can solve large size problems, but the measure is restricted to the compact region $[0,1]^n$ and this can cause problems. For instance, in the case of European call option, with change of numeraire, which will be introduced in the next section, we are able to consider the exit location measure upon two pieces of the support region $[0,1]$ and $[1,+\infty)$. It can be seen that we do not easily have linear moment conditions for the region $[1,+\infty)$. In addition, the moment conditions for LP are numerically ill-posed because of the binomial coefficients involved. Therefore, we prefer SDP formulations to bound the option prices. As to the convergence of the bounds with respect to different distributions as we increase knowing the number of moments, we refer to Lasserre et al [16]. In the next section, we will show applications of SDP formulations with change of numeraire. We will not only consider the European call option but also study exotic options such as exchange options with knowing up to fourth moment information.

2.3 Options with change of numeraire

In the case of one dimensional Black-Scholes formula, if there are two assets satisfying the following two diffusions

$$dS_t^0 = rS_t^0 dt, \quad dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dB_t^{\mathbb{P}},$$

where $S_t^0 = e^{rt}(S_0^0 = 1)$ denotes the price of a bond, S_t^1 denotes the price of a stock, μ and σ are constants in this paper so the Novikov condition $E[\exp\{\frac{1}{2}\int_0^T \sigma^2 ds\}] \leq \infty$ is automatically satisfied, and $B_t^{\mathbb{P}}$ is a Brownian motion on some filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. There exists a risk-neutral probability measure $\mathbb{Q}$ that is equivalent to probability measure $\mathbb{P}$, and such that discounted prices are martingales under $\mathbb{Q}$. We have the following well known lemma.

Lemma 2.3. *Let* $g(x) = (x - K)^+ = \max(x - K, 0)$. *The arbitrage free price of European call option can therefore be defined by*

$$C_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[g(S_T^1)] = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T^1 - K)^+], \tag{4.5}$$

The proof is well known and can be found in any text book of mathematical finance (see e.g. [9]). As we see from the lemma that the discounted stock process $\frac{S_t^1}{S_t^0}$ is a martingale under risk-neutral measure $\mathbb{Q}$ and so is the contingent claim $g(x)$. Because the stock prices are usually positive numbers ($x \in \mathbb{R}^+$), by employing the framework studied in section 2.1 the upper bound of problem (4.5) can actually be treated as the upper bound of the martingale measure supported on $[K, +\infty)$. By Theorem 2.2 (b), the moment and localizing matrices are required to be positive definite in order to be the sufficient moment conditions. This may cause numerical problems in solving SDPs as the constraints in SDP are usually positive semidefinite. However, for the problem (4.5) we observe that

$$C_0 = S_0\mathbb{E}^{\mathbb{Q}}(1 - \frac{K}{S_T})^+ \tag{4.6}$$

if we take the discounted stock out of the expectation and utilize the martingale property. The resulting upper bound problem becomes calculating the upper bound of the martingale measure supported on $[0, 1]$. Again, by Theorem 2.2 (b) the sufficient and necessary moment conditions only require the moment and localizing matrices to be positive semidefinite. The equation (4.6) can be formally obtained by taking the stock as the numeriaire. We have the following proposition.

Proposition 2.1. *If we let*

$$Y_t = \frac{S_t^0}{S_t^1},$$

where $S_T^0 = K$ *and* S_t^1, S_t^0 *satisfy*

$$dS_t^0 = rS_t^0 dt, \quad dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dB_t^{\mathbb{P}},$$

The arbitrage free price of European call option can therefore be defined by

$$C_0 = S_0^1 \mathbb{E}^{\mathbb{Q}'}[(1 - Y_T)^+], \tag{4.7}$$

where $Y_t = Y_0 \exp\{-\frac{1}{2}\sigma^2 t + \sigma B_t^{\mathbb{Q}'}\}$.

Proof. The proof is analogous to Lemma 2.3. By using Itô's Lemma and Girsanov's theorem, we obtain

$$dY_t = Y_t[(r - \mu + \sigma^2)dt - \sigma dB_t^{\mathbb{P}}] = \sigma Y_t dB_t^{\mathbb{Q}'},$$

from which we can deduce $Y_t = Y_0 \exp\{-\frac{1}{2}\sigma^2 t + \sigma B_t^{\mathbb{Q}'\}}$ that shows Y is a martingale and so is the call option C, and $Y_t = \frac{K}{S_0^1} \exp\{-(r + \frac{1}{2}\sigma^2 t) + \sigma B_t^{\mathbb{Q}'}\}$. Therefore, the European call option then can be defined as $C_0 = S_0^1 \mathbb{E}^{\mathbb{Q}'}[(1 - Y_T)^+]$. □

Moreover, in the case of computing bound on a interval $[a, b]$, Lasserre [7] shows that the bounds are discriminating tight when one has at least four moment conditions. With the change of numeraire, we are able to apply the problem of bounding option prices into this case and will show the bounds for the European call options and exchange options with up to fourth moments in the next section.

With regard to hedging strategies, we notice that a hedging strategy can be calculated via the dual formulation. The constraints of the dual formulation (D) show

$$\sum_{\alpha \in \mathbb{I}_d} \theta_\alpha x^\alpha \geq f(x).$$

If we take expectations on both sides, we obtain

$$\sum_{\alpha \in \mathbb{I}_d} \theta_\alpha \sigma_\alpha \geq E[f(x)],$$

which give us an over-hedged strategy with θ_0 in the cash bond and $\sum_{\alpha \in \mathbb{I}_d} \theta_\alpha \sigma_\alpha (\alpha \neq 0)$ in risky assets.

3 Applications and numerical results

We now illustrate applications of the proposed method to bound the options with up to fourth moment information. The SDPs are solved by GloptiPoly3 which is a Matlab/SeDuMi (see e.g. [15]) add-on solver available in public domain. The computations are done on a pentium IV 3.2G HZ PC with 1G RAM.

In the case of European call option, according to our numerical tests, solving the SDPs of bounding the European options without change of numeraire often runs into marginal feasible problem and can not get the global optimality certified numerically. This is, according to Theorem 2.2 , due to the positive definite requirements for the moment and localizing matrices when we are dealing with measures supported on unbounded region. We can, however, obtain the numerically certified global optimality using the stock as the numeraire which in fact changes the supported region of the objective measure to a bounded region. We compute the upper and lower bounds of a European call option (4.7) with knowing up to fourth moments of the martingale measure. For convenience of comparison, we assume the discounted stock is a exponential martingale as in the Black-Schole model with inputs such as

$$S_0 = 40, \quad r = 0.06, \quad \sigma = 0.2, \quad T = 1/52.$$

and employ the primal SDP formulation previously demonstrated. The results are shown in Table 1. The results are similar to [4] which utilizes the dual SDP formulation. It needs to be noted that one can increase the

Table 1: Bounds for European call option

	4-moments	3-moments	2-moments	
Strike	[UB, LB]	[UB, LB]	[UB, LB]	BS
30	10.0347 10.0346	10.0453 10.0346	10.0518 10.0346	10.0346
35	5.0419 5.0404	5.0768 5.0404	5.0866 5.0404	5.0404
40	0.5777 0.3422	0.5777 0.0461	0.5777 0.0461	0.4658
45	0.0042 0.0000	0.0773 0.0000	0.0773 0.0000	0.0000
50	0.0008 0.0000	0.0480 0.0000	0.0480 0.0000	0.0000

moments to attain tighter bounds, but the bounds, in this log-normal distribution case, will not converge to the exact value. This is pointed out by Lasserre et al [16] that only moment-determinate distributions can guarantee the convergence, and log-normal is not a moment-determinate distribution. The next numerical experiment is to calculate the tight bounds on European style exchange options which defined as

$$C_0 = \mathbb{E}^{\mathbb{Q}}[(S_T^1 - S_T^2)^+], \tag{4.8}$$

where $dS_t^i = S_t^i[\mu_i dt + \sigma_i dB_t^{\mathbb{P}}], \quad i = 1, 2$. Bounding the no arbitrage price of exchange options, at first glance, has two dimensions, we can have two pieces of the support region of the measure, one is $S_T^1 - S_T^2 \geq 0$ and the other is $S_T^1 - S_T^2 \leq 0$ on $\mathbb{R}^2$. The moments of exit location measure can be easily computed so that this problem can be solved by SDPs. Zuluaga and Pe$\tilde{n}$a [18] have computed the upper bound with first two moments of the measure supported on $\mathbb{R}_+^2$ in a similar manner. However, we note that this problem can be simplified to one dimension problem by using the change of numeraire. In fact, we actually have two numeraires to choose S_t^1 or S_t^2. If we choose S_t^2 as the numeraire, we obtain:

$$C_0 = S_0^2 \mathbb{E}^{\mathbb{Q}}[(\frac{S_T^1}{S_T^2} - 1)^+], \tag{4.9}$$

where $dY_t = Y_t(\sigma_1 - \sigma_2)dB_t^{\mathbb{Q}}, \quad Y_t = \frac{S_t^1}{S_t^2}$. In this case, we note that the two pieces of support region are $[1, \infty)$ and $(-\infty, 1]$ in $\mathbb{R}$ and the objective measure is supported on $[1, \infty)$. Similarly, if we take S_t^1 as the numeraire, we obtain:

$$C_0 = S_0^1 \mathbb{E}^{\mathbb{Q}}[(1 - \frac{S_T^2}{S_T^1})^+], \tag{4.10}$$

where $dY'_t = Y'_t(\sigma_2 - \sigma_1)dB_t^{\mathbb{Q}'}, \quad Y'_t = \frac{S_t^2}{S_t^1}$. In this case, we observe that the two pieces of support region are $[0, 1]$ and $[1, \infty)$ in $\mathbb{R}$ and the objective measure is supported on $[0, 1]$. Therefore, we prefer to choose S_t^1 as the numeraire because this change provides bounded region for the objective measure, and according to Theorem 2.2 we need only positive semidefinite moment and localizing matrices constraints to guarantee the sequences in the matrices are indeed the moments of some measure supported on the region. We take the inputs from Zuluaga and Peña [18]:

$$S_0^1 = 0.95, \quad S_0^2 = 0.90, \quad \sigma_1 = 0.2, \quad \sigma_2 = 0.22, \quad T = 1,$$

and the exact value is calculated from Margrabe [8]. We compute the upper and lower bounds with up to fourth moments. The results are shown in Table 2. We also can attain the lower bounds showing in Table 3. Note that we prefer to use S_T^1 as the numeraire as the objective measure is supported on a bounded region.

Table 2: Upper bounds for exchange options

		$\frac{S_T^2}{S_T^1}$	
ρ	Exact	2-mom	4-mom
-1.0	0.1801	0.2242	0.2114
-0.5	0.1600	0.1961	0.1888
0	0.1361	0.1641	0.1621
0.5	0.1051	0.1241	0.1240
1	0.0500	0.0516	0.0502

Table 3: Lower bounds for exchange options

		$\frac{S_T^2}{S_T^1}$	
ρ	Exact	2-mom	4-mom
-1.0	0.1801	0.0500	0.1233
-0.5	0.1600	0.0500	0.1152
0	0.1361	0.0500	0.1033
0.5	0.1051	0.0500	0.0844
1	0.0500	0.0500	0.0500

4 Conclusion

We have examined the applications of the moment approach on options pricing problems. The resulting problem can be cast and solved as SDP or LP problems. We prefer the SDP approach due to its simplicity of expressing the moment conditions on bounded and unbounded regions. We have proposed the technique of change of numeraire to bound option prices with moments constraints via SDP. This technique can, for the European call type options, change the supported region of the martingale measure from a semi-compact set to a compact set, which requires only positive semidefinite moment and localizing matrices rather than strict positive definite matrices as the sufficient moment conditions. This would solve the marginal feasible problems one often encounters when implementing the SDP models with larger inputs and higher moments. Moreover, it can simplify option pricing problems such as exchange options comparing with the method adopted by Zuluaga and Peña [18]. In this paper, we have only tested cases that are within the domain of $n \leq 2; d = 2; n = 3, d = 4$ and thus give us global optimal solutions. However, it is not hard to extend to other cases such as higher dimensions and higher moments with exploiting a hierarchy of SDP approximations.

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