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Portfolio Decisions with Higher Order Moments

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Abstract

In this paper, we address the global optimization of two interesting nonconvex problems in finance. We relax the normality assumption underlying the classical Markowitz mean-variance portfolio optimization model and consider the incorporation of skewness (third moment) and kurtosis (fourth moment). The investor seeks to maximize the expected return and the skewness of the portfolio and minimize its variance and kurtosis, subject to budget and no short selling constraints. In the first model, it is assumed that asset statistics are exact. The second model allows for uncertainty in asset statistics. We consider rival discrete estimates for the mean, variance, skewness and kurtosis of asset returns. A robust optimization framework is adopted to compute the best investment portfolio maximizing return, skewness and minimizing variance, kurtosis, in view of the worst-case asset statistics. In both models, the resulting optimization problems are nonconvex. We introduce a computational procedure for their global optimization.

Keywords: Mean-variance portfolio selection, Robust portfolio selection, Skewness, Kurtosis, Decomposition methods, Polynomial optimization problems

1 Introduction

In this paper, we consider two interesting finance applications. Both are extensions of well-established convex models to their nonconvex counterpart. The first finance application we consider is the problem of selecting an optimal investment portfolio that consists of holdings in a number of assets, assuming that the asset statistics are exact. According to the classical

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mean-variance approach devised by [1] the investor's goal is to maximize the expected return of the portfolio (*first moment* or *mean*) and minimize its risk (*second central moment* or *variance*). However, the aforesaid model is based on the assumption that asset returns are normally distributed. As empirical evidence suggests [2], normality may not be the case in reality. On the contrary, asset return distributions are generally characterized by asymmetries and/or fat tails [3, 4]. In order to relax the normality assumption, we incorporate *skewness* (*third central moment*) and *kurtosis* (*fourth central moment*) in the optimal portfolio selection. In our model the investor's goal is to maximize the expected return and the skewness of the portfolio, and minimize the variance and the kurtosis of the portfolio, subject to satisfying the budget constraint and excluding short sales. Our choice is supported by the generally established fact that investors prefer odd moments and are averse to the even ones [5][1].

The consideration of higher moments in portfolio selection is in fact a very old idea. Since at least early sixties there has been a controversy over the issue whether or not higher order moments should be incorporated into the portfolio selection. Some studies, such as [7, 8, 9, 10, 11, 12], supported the importance of higher moments in optimal portfolio selection, and others, like [13, 14, 15, 16], have regarded the consideration of higher order moments with disfavor[2]. However, almost all recent studies suggest that significant gains and great potential arise from taking into account higher moments [17, 18, 19, 20, 21, 22, 23]. The latter studies have offered a substantial insight into the resulting nonconvex portfolio selection problem, as well as the corresponding three-dimensional *efficient frontiers*. Nonetheless, their solution strategies see the problem from local optimization viewpoint[3], e.g. Lai and Chunhachinda et al. employ polynomial goal programming, Athayde et al. and Jondeau et al. use first order conditions. Our work also adopts the belief that higher moments should not be neglected, but differs from the foregoing works in that it formulates and solves the (nonconvex) portfolio selection problem in a general global optimization framework. A closely related work has been carried out by Parpas et al. who apply a stochastic global optimization algorithm to solve the nonconvex portfolio selection problem [24]. The interested reader is also referred to the subsequent work by Maringer et al. [25]. As far as the general use of global optimization

[1]Brockett et al. [6] contradict the fact that investors prefer the odd to the even moments, but such a discussion is out of the scope of this thesis.

[2]These authors consolidated the adequacy of the mean-variance approximations for various utility functions and empirical return distributions.

[3]These authors employ local approaches to solve the global optimization problem arising.

in finance is concerned, Konno provides a review of global optimization in portfolio selection models, and Maranas et al. use a deterministic global optimization algorithm to tackle a multi-period model [26].

The second finance application considered in this paper is the robust counterpart of the mean-variance model. In this analysis, the asset statistics are not assumed to be exact as was the case before. As a result, the portfolio return and risk are expressed by the worst-case mean and variance of the portfolio, respectively. By introducing uncertainty to the skewness and kurtosis estimates in addition to mean and variance estimates and assuming the existence of discrete rival asset estimates, we investigate the robust mean-variance-skewness-kurtosis portfolio optimization problem. Although the incorporation of higher moments into portfolio selection has been considered by several authors, as pointed out above, the only work, to the best of our knowledge, investigating its robust counterpart is by Harvey et al. [21], but from a significantly different perspective. The authors in that work treat the portfolio selection problem with higher moments as a two-stage problem. At the first stage they employ a Bayesian probability model to deal with the data uncertainty. At the second stage they maximize the mean-variance-skewness expected utility function for the exact asset estimates, which are the output of the first stage. So, even though the parameter uncertainty is not ignored, their model does not formally fall into the well-known robust framework. On the other hand, our work deals, for the first time, with data uncertainty and higher moments in a robust global optimization framework.

The computational procedure, that we propose, is applicable to polynomial optimization problems. A polynomial optimization problem is defined as the problem of finding the minimum of a real-valued multivariate polynomial $p(x) : \mathbb{R}^n \to \mathbb{R}$, either unconstrained or constrained in a compact set $\mathcal{K}$ defined by polynomial inequalities and equalities:

$$p^* = \min_{x \in \mathcal{K}} p(x). \tag{1}$$

A set, such as $\mathcal{K}$, comprised of polynomial inequalities and equalities is called *basic closed semialgebraic*. Polynomial optimization problems, also known as POPs, are global optimization problems and are of great theoretical and practical importance. For the interested reader, the global optimization of polynomials is tackled in [27, 28, 29]. We employ the results from these works into the context of a decomposition-based algorithm, which we apply to the resulting polynomial optimizations problems arising from both finance applications.

Contribution. Our contribution can be summarized as follows: we reformulate two well-established (convex) finance models to their nonconvex counterpart by including higher order moments. We tackle the resulting models in a global optimization framework by employing a decomposition scheme made for polynomial optimization.

This paper is organized as follows: In Section 2, we address the portfolio optimization problem with skewness and kurtosis. In Section 3, the worst-case mean-variance-skewness-kurtosis problem for discrete uncertainty sets is modelled. In Section 4, we describe the global optimization algorithm that we employ to solve the resulting class of problems. In Section 5, we solve both models, for several assets and investor's preferences, with the proposed method and present the numerical results. Section 6 recapitulates.

Notation. The notation adopted in the entire paper is as follows: R_{it} denotes the return on asset i at time t and N the total number of returns on asset i. In addition, R_i expresses the average return on asset i. Next, let μ_i be the expected return (mean) of R_i and σ_{ij} be the covariance between R_i and R_j. Similarly, let s_{ijk} be the coskewness of R_i, R_j and R_k and k_{ijkl} the cokurtosis of R_i, R_j, R_k and R_l. It is clear that σ_{ii}, s_{iii}, k_{iiii} are the variance, skewness and kurtosis of R_i, respectively. These asset statistics and their formulae are summarized in Tables 1 and 2, respectively. In a portfolio consisting of n assets, the collection of mean estimates $\mu_1, \ldots, \mu_n$ forms the vector of means $\mu \in \mathbb{R}^n$, and the variance/covariance estimates form the covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$. In the same vein, the skewness/coskewness estimates and the kurtosis/cokurtosis estimates are elements of the coskewness matrix $S \in \mathbb{R}^{n\times n^2}$ and cokurtosis matrix $K \in \mathbb{R}^{n\times n^3}$, respectively. For example, in a portfolio with two assets, i.e. $n = 2$, the asset estimates read:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \quad S = \begin{bmatrix} s_{111} & s_{112} & s_{211} & s_{212} \\ s_{121} & s_{122} & s_{221} & s_{222} \end{bmatrix}$$

Moment	**Symbol**
Expected return (Mean) of asset *i*	$\boldsymbol{\mu_i}$
Variance of asset *i*	$\boldsymbol{\sigma_{ii}}$
Skewness of asset *i*	$\boldsymbol{s_{iii}}$
Kurtosis of asset *i*	$\boldsymbol{k_{iiii}}$
Covariance of assets *i* and *j*	$\boldsymbol{\sigma_{ij}}$
Coskewness of assets *i*, *j* and *k*	$\boldsymbol{s_{ijk}}$
Cokurtosis of assets *i*, *j*, *k* and *l*	$\boldsymbol{k_{ijkl}}$

Table 1: Asset Statistics (Moments): Symbols

Moment	**Definition**	**Formula**
$\boldsymbol{\mu_i}$	$E[R_i]$	$\frac{1}{N}\sum_{t=1}^{N} R_{it}$
$\boldsymbol{\sigma_{ii}}$	$E[(R_i-\mu_i)^2]$	$\frac{1}{N-1}\sum_{t=1}^{N}(R_{it}-\mu_i)^2$
$\boldsymbol{s_{iii}}$	$E[(R_i-\mu_i)^3]$	$\frac{1}{N}\sum_{t=1}^{N}(R_{it}-\mu_i)^3$
$\boldsymbol{k_{iiii}}$	$E[(R_i-\mu_i)^4]$	$\frac{1}{N}\sum_{t=1}^{N}(R_{it}-\mu_i)^4$
$\boldsymbol{\sigma_{ij}}$	$E[(R_i-\mu_i)(R_j-\mu_j)]$	$\frac{1}{N-1}\sum_{t=1}^{N}(R_{it}-\mu_i)(R_{jt}-\mu_j)$
$\boldsymbol{s_{ijk}}$	$E[(R_i-\mu_i)(R_j-\mu_j)(R_k-\mu_k)]$	$\frac{1}{N}\sum_{t=1}^{N}(R_{it}-\mu_i)(R_{jt}-\mu_j)(R_{kt}-\mu_k)$
$\boldsymbol{k_{ijkl}}$	$E[(R_i-\mu_i)(R_j-\mu_j)(R_k-\mu_k)(R_l-\mu_l)]$	$\frac{1}{N}\sum_{t=1}^{N}(R_{it}-\mu_i)(R_{jt}-\mu_j)(R_{kt}-\mu_k)(R_{lt}-\mu_l)$

Table 2: Asset Statistics (Moments): Definitions & Formulae

and

$$K = \begin{bmatrix} k_{1111} & k_{1112} & k_{1211} & k_{1212} & k_{2111} & k_{2112} & k_{2211} & k_{2212} \\ k_{1121} & k_{1122} & k_{1221} & k_{1222} & k_{2121} & k_{2122} & k_{2221} & k_{2222} \end{bmatrix}.$$

2 Portfolio Optimization with Skewness & Kurtosis

We consider a portfolio of n risky assets held over a single period. The profit R on the portfolio as whole is $R = \sum_{i=1}^{n} x_i R_i$, where x_i is the proportion of the portfolio invested on asset i. Observe that the R_i's, and consequently R, are random variables. Hence, the return R of the portfolio is a weighted sum of random variables. According to the classical mean-variance model [1], the investor originally sought to find the weights so as to maximize his expected profit (mean of portfolio return) with the minimum possible risk (variance of portfolio return). The optimization problem that arises from this approach is quadratic and has several equivalent formulations. The reader may consult [30] for a detailed review. We present two of these below:

$$\max_{x \in X} \lambda_1 \sum_{i=1}^{n} \mu_i x_i - \lambda_2 \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j, \tag{2}$$

$$\begin{aligned} \min_{x \in X} \quad & \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j \\ \text{s.t.} \quad & \sum_{i=1}^{n} \mu_i x_i \geq R_{\min}. \end{aligned} \tag{3}$$

The input parameter $R_{\min}$ on the right hand side of the constraint in the latter model represents a lower bound on the expected return. On the other hand, the input parameters λ_1 and λ_2 in the former formulation sum up to one, i.e. $\lambda_1 + \lambda_2 = 1$, and express the investor's preferences towards the importance of mean and variance. Both formulations model the trade-off between the expected return and the risk. By solving problem (2) for different values of (λ_1, λ_2), or problem (3) for different values of $R_{\min}$, one can obtain a sequence of optimal portfolios on the so-called *efficient frontier*[4]. In both models, X represents the set of *feasible portfolios*. The x_i's are percentages and not the actual amount invested on each asset, and as a result we have the constraint $\sum_{i=1}^{n} x_i = 1$ to represent the budget constraint. In addition, short sales are excluded, so we have that $x_i \geq 0$ for all i. These two types of constraints form a *polyhedral* set of feasible portfolios:

$$X = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1,\ x \geq 0\}. \tag{4}$$

In our analysis, the goal of the investor is a generalization of the goal in (2). In particular, the investor aims at finding the weights so as to maximize his odd moments (mean, skewness) while minimizing his even moments (variance, kurtosis). The portfolio weights are still constrained in the set X. We also assume that the estimates of asset statistics are exact. Thus, for fixed

[4] An *efficient portfolio* provides the maximum expected return for a given variance or less, and the minimum variance for a given expected return or more [31].

asset statistics $\{\mu_i\}, \{\sigma_{ij}\}, \{s_{ijk}\}$ and $\{k_{ijkl}\}$ the problem of choosing the optimal portfolio based on the first four moments of the portfolio return becomes:

$$\max_{x \in X} \lambda_1 \sum_{i=1}^{n} \mu_i x_i - \lambda_2 \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j + \lambda_3 \sum_{i,j,k=1}^{n} s_{ijk} x_i x_j x_k - \lambda_4 \sum_{i,j,k,l=1}^{n} k_{ijkl} x_i x_j x_k x_l. \quad (5)$$

In the same vein as in (2), the scalars λ_1 to λ_4 are the investor's preferences to the four moments and they sum up to one, i.e. $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$. The objective function in formulation (5) is a real-valued polynomial of degree four, the objective vector is $x \in \mathbb{R}^n$ and the set X is a simplex. It is clear that (5) is a polynomial optimization problem of total degree four.

3 Robust Portfolio Optimization with Skewness & Kurtosis

Contrary to the previous section, the asset statistics are not assumed to be exact in the analysis that follows. In particular, we assume initially that uncertainty underlies our knowledge of the mean and variance/covariance estimates. Let $\mathcal{U}_\mu$ and $\mathcal{U}_\Sigma$ denote the uncertainty sets the mean vector μ and the covariance matrix Σ belong to, respectively. In general, the latter sets can represent a finite number of scenarios, i.e. discrete mean and variance/covariance estimates, or they can be interval-type, or ellipsoidal uncertainty sets [32, p. 293]. In order to remain in a polynomial optimization framework, we assume our uncertainty sets are discrete. In a portfolio selection problem, based on the traditional Markowitz approach, the goal of the investor verbally remains the same: he or she seeks to minimize the portfolio risk, subject to a lower bound on expected return, and subject to budget and no short selling constraints. However, in this case the portfolio moments are expressed by their worst-case analogues:

$$\min_{\mu \in \mathcal{U}_\mu} \sum_{i=1}^{n} \mu_i x_i, \quad (6)$$

$$\max_{\Sigma \in \mathcal{U}_\Sigma} \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j. \quad (7)$$

Observe that in the formulations above μ and Σ are the objective variables, while the portfolio weights $(x_1, \ldots, x_n)$ are considered fixed. By incorporating the worst-case portfolio mean (6) and worst-case portfolio variance (7) into the portfolio selection problem (2), or (3), the robust mean-variance portfolio selection problem reads:

$$\max_{x\in X} \min_{\substack{\mu\in\mathcal{U}_\mu \\ \Sigma\in\mathcal{U}_\Sigma}} \lambda_1 \sum_{i=1}^{n} \mu_i x_i - \lambda_2 \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j, \tag{8}$$

or,

$$\begin{aligned} &\min_{x\in X} \max_{\Sigma\in\mathcal{U}_\Sigma} && \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j \\ &\text{s.t.} && \min_{\mu\in\mathcal{U}_\mu} \sum_{i=1}^{n} \mu_i x_i \geq R_{\min}, \end{aligned} \tag{9}$$

where X is given in (4). The models (8) and (9) are motivated by [33, 34, 35, 36, 37].

In what follows we assume uncertainty not only in our knowledge of the mean vector and the covariance matrix, but also in our knowledge of the coskewness and cokurtosis matrices. Let $\mathcal{U}_S$ and $\mathcal{U}_K$ denote the uncertainty sets that coskewness and cokurtosis matrices belong to. The worst-case analogues of portfolio skewness and portfolio kurtosis are:

$$\min_{S\in\mathcal{U}_S} \sum_{i,j,k=1}^{n} s_{ijk} x_i x_j x_k \tag{10}$$

$$\max_{K\in\mathcal{U}_K} \sum_{i,j,k,l=1}^{n} k_{ijkl} x_i x_j x_k x_l. \tag{11}$$

Our generalized goal remains verbally the same as the goal in (5), namely to minimize the portfolio risk expressed by the even moments, while maximizing the odd moments, subject to budget and no short selling constraints. Hence, our model for discrete uncertainty sets is:

$$\max_{x\in X} \min_{\substack{\mu\in\mathcal{U}_\mu,\Sigma\in\mathcal{U}_\Sigma \\ S\in\mathcal{U}_S,K\in\mathcal{U}_K}} \lambda_1 \sum_{i=1}^{n} \mu_i x_i - \lambda_2 \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j + \lambda_3 \sum_{i,j,k=1}^{n} s_{ijk} x_i x_j x_k - \lambda_4 \sum_{i,j,k,l=1}^{n} k_{ijkl} x_i x_j x_k x_l. \tag{12}$$

Problem (12) is a max-min optimization problem, which translates trivially into a polynomial optimization problem. Namely, by introducing four scalars, one for each portfolio moment, we reformulate the above problem as follows:

$$
\begin{array}{llrcl}
\max\limits_{\substack{x\in X \\ z_1,z_2,z_3,z_4}} & \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4 & & & \\
\text{s.t.} & \sum_{i=1}^{n} \mu_i^{(k_1)} x_i & \geq & z_1, & k_1 = 1,\ldots,|\mathcal{U}_\mu|, \\
& -\sum_{i,j=1}^{n} \sigma_{ij}^{(k_2)} x_i x_j & \geq & z_2, & k_2 = 1,\ldots,|\mathcal{U}_\Sigma|, \\
& \sum_{i,j,k=1}^{n} s_{ijk}^{(k_3)} x_i x_j x_k & \geq & z_3, & k_3 = 1,\ldots,|\mathcal{U}_S|, \\
& -\sum_{i,j,k,l=1}^{n} k_{ijkl}^{(k_4)} x_i x_j x_k x_l & \geq & z_4, & k_4 = 1,\ldots,|\mathcal{U}_K|,
\end{array}
\tag{13}
$$

where the notation $|\mathcal{U}_\cdot|$ refers to the number of discrete scenarios belonging to each uncertainty set. The class of problems resulting from (13) can be treated into the polynomial optimization framework as these are quartic polynomial optimizations problems.

4 Algorithm: Partitioning Procedure for Polynomial Optimization

Notation: By $\mathbb{R}[x] = \mathbb{R}[x, \ldots, x_n]$ we denote the *polynomial ring* over $\mathbb{R}$ in n variables. In addition, we use $\Sigma^2 \subseteq \mathbb{R}[x]$ to denote the set of squares of polynomials in this polynomial ring.

We consider the following polynomial optimization problem (POP):

$$
\begin{array}{rll}
p^* = & \min\limits_{x,y} & p(x,y) \\
& \text{s.t.} & g_i(x,y) \geq 0,\ i = 1,\ldots,m, \\
& & h_j(x,y) = 0,\ j = 1,\ldots,p, \\
& & x \in X,\ y \in Y,
\end{array}
\tag{14}
$$

where $p, g_1, \ldots, g_m, h_1, \ldots, h_p \in \mathbb{R}[x]$. Also, $x = (x, y) \in \mathbb{R}^n$ and the sets $X \subseteq \mathbb{R}^{n_1}$ and $Y \subseteq \mathbb{R}^{n_2}$, where $n = n_1 + n_2$, are assumed to be convex and compact. The feasible region of our problem is a basic closed semialgebraic set, namely a set of polynomial inequalities and equalities, assumed non-empty and compact:

$$
\mathcal{K} = \{(x,y) \in X \times Y \subseteq \mathbb{R}^n \mid g_i(x,y) \geq 0,\ \forall i,\ h_j(x,y) = 0,\ \forall j\}. \tag{15}
$$

POPs are generally characterized by nonconvexities, hence are global optimization problems. As has been shown, one is able to convexify a POP by employing the moment problem and its interaction with positive polynomials and semidefinite programming [27, 28]. In particular, one can approximate p^* by solving a sequence of (convex) semidefinite (SDP) relaxations of increasing size. The relaxations can be solved efficiently by interior-point methods in polynomial time [38]. The solutions of the relaxations provide lower bounds to the global optimal solution p^* of the POP. These bounds converge asymptotically to p^* [27]. However, the size of the POPs tackled by this SDP relaxation technique is limited.

Decomposition methods have always found application in mathematical programming when one tackled a large-scale problem. These methods convert the solution of the original problem into the solution of a series of problems of lower dimension. For this reason, we aim at tackling POPs using decomposition. To achieve this, we extend the well-known generalized Benders decomposition for convex programs [39] to the global optimization of polynomials by employing the powerful theoretical results underlying the SDP relaxation technique. This technique is described in detail in [29].

4.1 Derivation of the Master Problem

The essence of the generalized Benders decomposition is to initially derive the so-called *master problem* such that it is equivalent to the original problem, and secondly employ a series of subproblems in order to solve the master problem.

If we apply the concept of projection [40], often referred to as partitioning, we can express problem (14) as a problem in y-space as follows.

$$\begin{aligned} p^* = \ & \min_{y} && v(y) \\ & \text{s.t.} && y \in Y \cap V, \end{aligned} \tag{16}$$

where

$$\begin{aligned} v(y) = \ & \inf_{x \in X} && p(x,y) \\ & \text{s.t.} && g_i(x,y) \geq 0, \ i = 1,\ldots,m, \\ & && h_j(x,y) = 0, \ j = 1,\ldots,p, \end{aligned} \tag{17}$$

and

$$V = \{y \mid g_i(x,y) \geq 0,\ \forall i,\ h_j(x,y) = 0,\ \forall j, \text{ for some } x \in X\}. \tag{18}$$

Observe that $v(y)$ is the optimal value of (14) for fixed y. Hence, $v(y)$ is an upper bound on p^*. To obtain $v(y)$ for fixed y we have to solve the inner POP,

$$\begin{aligned} \min_{x\in X} \quad & p(x,y) \\ \text{s.t.} \quad & g_i(x,y) \geq 0,\ i = 1,\dots,m, \\ & h_j(x,y) = 0,\ j = 1,\dots,p. \end{aligned} \tag{19}$$

The set V introduced earlier consists of those values of y for which (19) is feasible and $Y \cap V$ is the projection of the feasible region of (14) onto y-space. Therefore, by projection we managed to express problem (14) as a problem onto y-space, namely in terms of problem (16). Problem (16) is equivalent to (14) and it is the route to solving it [40, Theorem 1]. According to the generalized Benders decomposition, projection is the first of the three problem manipulatins that are required to derive the master problem. The next two manipulations consist of invoking the dual representations of V and $v(y)$. To implement these manipulations, we employ the Theorems 3.2 and 3.3 from [41], respectively, and reformulate problem (16) as follows:

$$\begin{aligned} \min_{y,z} \quad & z \\ \text{s.t.} \quad & 0 \geq \inf_{x\in X}\{-\sum_{I\subseteq\{1,\dots,m\}} \sigma_I(x) g_I(x,y) - \sum_{j=1}^{p} t_j(x) h_j(x,y)\},\ \forall \sigma_I \in \Sigma^2,\ t \in \mathbb{R}[x], \\ & z \geq \inf_{x\in X}\{p(x,y) - \sum_{i=0}^{m} \sigma_i(x) g_i(x,y) - \sum_{j=1}^{p} t_j(x) h_j(x,y)\},\ \forall \sigma_i \in \Sigma^2,\ t \in \mathbb{R}[x], \end{aligned} \tag{20}$$

which is equivalent to (14) and is our master problem. Theorem 3.2 in its turn employs the Positivstellensatz [42] to express conditions that prevent the semialgebraic set of the parametrized subproblem (19) from being empty. On the other hand, Theorem 3.3 is applied when this set is nonempty, i.e. feasible, and expresses valid inequalities, by employing Theorem 4.2 from [27], in order to cut off suboptimal points from the feasible set. As a result, we obtain the set of the so-called feasibility and optimality constraints. These two types of constraints appear in the master problem (20) in the first and second row, respectively. However, the number of constraints in the master problem (20) is infinite. For this reason *relaxation* is followed as a solution strategy [40]. In other words, we begin by solving a relaxed version of (20), the so-called *relaxed master problem*, ignoring all but few constraints and if the resulting solution does not satisfy all of the ignored constraints we generate and add to the relaxed master problem one violated constraint (either from the set of feasibility constraints or from the set of optimality constraints). We continue this way until a termination criterion is satisfied which signals that the obtained solution is optimal within an acceptable accuracy. The equivalence of the master problem to the original POP implies that every time we solve a relaxed version of the master problem we get a lower bound on the optimal value of (14). Hence, solving a series of relaxed master problems yields a sequence of monotonically increasing lower bounds on the global optimal solution p^*. The algorithm is summarized in Figure 1. The interested reader is referred to [41] for more details and theoretical results. In this work, we prove in Theorem 3.4 that our procedure terminates without cycling and attains ϵ-global optimality. Moreover, asymptotic ϵ-convergence of our procedure is shown in Theorem 3.5. The asymptoticity comes from the underlying SDP relaxation technique. Nevertheless, practice demonstrated that the algorithm generally terminates in a finite number of iterations. Finally, we test the performance of our algorithm on a collection of benchmark problems from GlobalLib [43].

5 Numerical Results

Our data set includes historical stock prices obtained from `uk.finance.yahoo.com`. The stocks considered are the stocks that form the *Dow Jones Industrial Average*, also called the *Dow 30*. The historical prices in our possession cover the period between 2 April 1990 and 3 May 2006 on a monthly basis. Let us denote two successive historical prices as $P_{i,t}$ and $P_{i,t+T}$, where $i = 1, \dots, n$, $t = 1, \dots, N+1$ and T is a period of one month, then the asset return R_{it} corresponding to asset i and time period t is:

$$R_{it} = \frac{P_{i,t+T} - P_{i,t}}{P_{i,t}}. \tag{21}$$

After converting the historical prices into asset returns using (21), the formulae presented in Table 2 were employed to compute the asset statistics μ_i, σ_{ij}, s_{ijk}, k_{ijkl}, for all $i, j, k, l =$

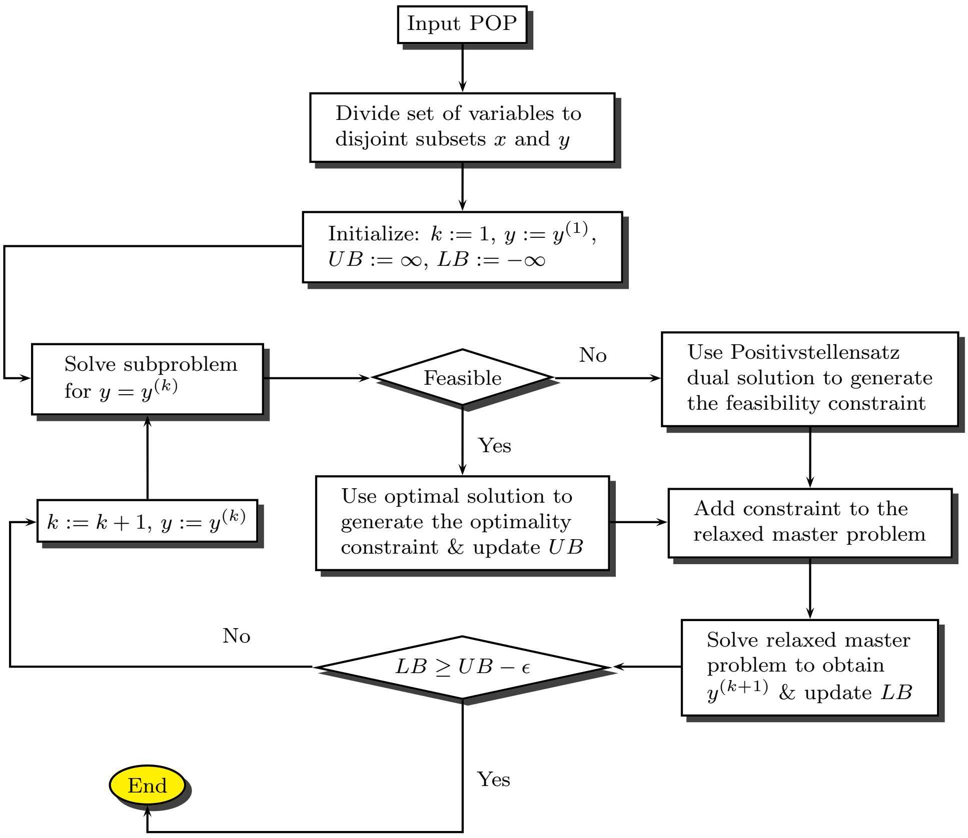


Figure 1: Partitioning procedure for POPs

$1, \ldots, n$. In the robust case, where the asset statistics belong to a discrete set of rival estimates, we perturbed the historical prices in possession to derive a different collection of prices. This perturbation process was performed as many times as the number of scenarios[5]. Next, for each different collection of prices we computed the asset statistics using again Equation (21) and the formulae from Table 2. In Table 3, the computed asset statistics (moments) of the original data are presented[6]. Note that the comoments such as covariance, coskewness and cokurtosis, are not presented for the sake of a convenient presentation. For randomly generated investor's preferences $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ (see Table 4) and for several combinations of stocks we created a number of models based on Equation (5). In addition, for several scenarios we created the corresponding worst-case models based on Equation (13). Table 5 summarizes the name and notation we use for each problem instance[7]. In what follows, MVO stands for Mean-Variance Optimization corresponding to the model (2). Similarly, MVSKO and RMSKO stand for Mean-Variance-Skewness-Kurtosis Optimization, i.e. model (5), and Robust Mean-Variance-Skewness-Kurtosis Optimization, i.e. model (13), respectively.

[5] The number of rival scenarios was the same for the four portfolio moments, i.e. $k_1 = \ldots = k_4$ in equation (13).

[6] The stock of KRAFT FOODS INC (KFT) was the only stock, among the components of the Dow 30, with no historical data available since as early as April 1990, and it was left out of our portfolios.

[7] The problems were written in GAMS scalar format.

	Mean	Variance	Skewness	Kurtosis		Mean	Variance	Skewness	Kurtosis
3M	0.004197	0.006159	−0.001264	0.0007842	**Intel**	0.008035	0.02043	−0.002743	0.002066
Alcoa	0.003997	0.01248	−0.001289	0.001574	**IBM**	0.004121	0.01031	−0.000572	0.000746
Amex	0.008432	0.008001	−0.001687	0.001056	**J & J**	0.005288	0.008085	−0.001944	0.001119
AT&T	0.0003379	0.007329	−0.0008968	0.0006125	**JPMorgan**	0.01004	0.01216	−0.0008873	0.0008576
Bank of America	0.006625	0.009463	−0.001024	0.0008125	**Kraft Foods**	–	–	–	–
Boeing	0.005623	0.007976	−0.001	0.0005436	**McDonald's**	0.005275	0.007538	−0.00137	0.0007816
Caterpillar	0.007621	0.01105	−0.001484	0.001165	**Merck**	0.001962	0.009355	−0.001836	0.001284
Chevron	0.003073	0.005566	−0.001118	0.0006448	**Microsoft**	0.005384	0.01749	−0.002078	0.001977
Cisco	0.01493	0.02657	−0.003976	0.003438	**Pfizer**	0.003672	0.01148	−0.003261	0.002153
Coca-Cola	0.00123	0.006478	−0.001302	0.0006626	**P & G**	0.004114	0.008042	−0.002227	0.001241
DuPont	0.003854	0.005465	−0.0003252	0.0002285	**Travelers**	0.005575	0.009206	−0.0003556	0.001086
ExxonMobil	0.004853	0.004799	−0.00123	0.0006995	**United Tech.**	0.007094	0.009772	−0.002669	0.001558
General Electric	0.002714	0.008691	−0.00249	0.001596	**Verizon**	−0.001229	0.006466	−0.0007095	0.000707
Hewlett-Packard	0.007825	0.01632	−0.001859	0.001692	**Wal-Mart**	0.005395	0.009374	−0.001814	0.001116
The Home Depot	0.005372	0.01055	−0.001282	0.0008179	**Walt Disney**	0.001455	0.01133	−0.003588	0.002773

Table 3: Moments of assets used

Model	λ_1	λ_2	λ_3	λ_4	Model	λ_1	λ_2	λ_3	λ_4
1	0.267	0.256	0.252	0.225	13	0.196	0.019	0.293	0.492
2	0.377	0.0601	0.395	0.168	14	0.27	0.118	0.2	0.411
3	0.276	0.25	0.213	0.261	15	0.391	0.498	0.0994	0.0118
4	0.308	0.0697	0.396	0.226	16	0.43	0.123	0.114	0.333
5	0.255	0.15	0.365	0.23	17	0.125	0.155	0.612	0.108
6	0.21	0.132	0.378	0.28	18	0.426	0.246	0.0177	0.31
7	0.0739	0.255	0.298	0.374	19	0.218	0.437	0.0861	0.258
8	0.548	0.0597	0.113	0.279	20	0.564	0.157	0.233	0.0465
9	0.248	0.257	0.268	0.227	21	1.000	0.000	0.000	0.000
10	0.332	0.287	0.0541	0.326	22	0.000	1.000	0.000	0.000
11	0.243	0.301	0.295	0.161	23	0.000	0.000	1.000	0.000
12	0.0488	0.489	0.0212	0.441	24	0.000	0.000	0.000	1.000

Table 4: Investor's preferences: trade-off among four portfolio moments

Model	Type	Assets	Scenarios	Problem Name
i	MVSKO	n	–	portfolio$i$_n
i	RMVSKO	n	k	portfolio$i$_n_k

Table 5: Summary of Problem Instances

Table 6 contains the optimal portfolios for each problem instance after we applied the partitioning procedure discussed in Section 4. In particular, the optimal vector x of portfolio weights[8] (multiplied by ten), and the optimal values of the four portfolio moments are reported in the first five columns of the table. The last column holds the number of iterations performed by our algorithm such that a 10^{-8} accuracy between the lower and upper bounds computed is achieved[9]. The missing values in a row denote that the specific instance was not handled by our program. For further numerical results, the reader may consult Table 5 in [41]. Table 6 in this paper and Table 5 in [41] demonstrate that the optimization of the first four moments of a portfolio consisting of up to twenty assets, in the deterministic case, and up to up to sixteen assets and ten scenarios, in the robust max-min case, can be solved efficiently by our procedure. Finally, Figures 2 and 3 depict, respectively, the resulting efficient frontiers and lines for three, six and ten assets[10]. Observe that the robust efficient frontier (RMVSKO)

[8]Note that all reported portfolio weights add up to one, but due to rounding may not appear to do so.

[9]The accuracy was decreased to 10^{-6} for portfolios with more than six assets. For this reason, problems of smaller size may appear to require more iterations than larger problems.

[10]The portfolio of three assets includes the Cisco, J & J and JPMorgan stocks. The portfolio of six assets includes the Alcoa, Amex, Caterpillar, Cisco, General Electric and Hewlett-Packard stocks. The portfolio of ten assets includes the 3M, Amex, Boeing, Cisco, General Electric, Home Depot, Intel, J & J, Microsoft and

is a lower bound on the classical efficient frontier (MVO) in Figure 2. This in line with the results in [44] where the robust MVO is shown to yield an efficient frontier lower or equal to the infimum of all classical efficient frontiers consistent with the model. In particular, when $\mathcal{U}_\mu$ and $\mathcal{U}_\Sigma$ are convex the robust efficient frontier coincides with the infimum of all sampled efficient frontiers [44]. Based on our empirical findings, this fact appears to be also true in the efficient M-V-S and M-V-K lines in Figure 3. On the other hand, marginally lower than the classical MVO frontier is the MVSKO efficient frontier in Figure 2. This should be justified by the not so large departure from normality of the selected assets. Jondeau et al. support that using randomly selected US stocks is not always appropriate [22]. For this reason, among other data sets, they use three specific former components of the index S&P 100 because these are characterized by large departure from normality[11]. Taking this into account, more tests for several data sets is on progress.

Wal-Mart stocks.

[11] Due to changes in the market, we were not able to retrieve the historical prices for these three stocks.

Problem	$\mathbf{10 * x}$	***P. Mean***	***P. Var***	***P. Skew***	***P. Kurt***	**Iters**
portfolio1_6	$(0.0, 2.0, 4.8, 0.0, 1.6, 1.6)$	0.0104	0.0965	-0.0764	0.1326	16
portfolio1_6_10	$(1.6, 2.1, 3.2, 0.0, 2.0, 1.1)$	0.0086	0.1061	-0.0568	0.1021	$5\frac{1}{2}$
portfolio2_6	$(0.0, 0.8, 6.6, 0.0, 0.5, 2.0)$	0.0123	0.1226	-0.1058	0.1724	9
portfolio2_6_10	$(0.6, 1.5, 4.8, 0.0, 1.8, 1.3)$	0.0103	0.1347	-0.0765	0.1315	$4\frac{1}{2}$
portfolio3_6	$(2.3, 2.7, 2.0, 0.2, 1.8, 0.9)$	0.0073	0.0633	-0.0498	0.0875	21
portfolio3_6_10	$(2.2, 2.4, 1.4, 1.3, 1.8, 0.9)$	0.0064	0.0824	-0.0465	0.0815	$6\frac{1}{2}$
portfolio4_6	$(1.9, 2.6, 2.4, 0.0, 1.9, 1.1)$	0.0078	0.0676	-0.0509	0.0925	20
portfolio4_6_10	$(2.0, 2.5, 1.6, 1.1, 1.8, 0.9)$	0.0067	0.0846	-0.0468	0.0831	$5\frac{1}{2}$
portfolio5_6	$(1.4, 2.7, 3.3, 0.0, 1.6, 1.0)$	0.0087	0.0760	-0.0600	0.1043	16
portfolio5_6_10	$(2.0, 2.4, 1.9, 0.9, 1.9, 0.9)$	0.0070	0.0870	-0.0478	0.0849	$3\frac{1}{2}$
portfolio6_6	$(2.9, 2.6, 0.5, 1.8, 1.7, 0.6)$	0.0053	0.0542	-0.0478	0.0786	21
portfolio6_6_10	$(2.5, 2.2, 0.9, 1.7, 1.9, 0.9)$	0.0057	0.0788	-0.0464	0.0793	$4\frac{1}{2}$
portfolio7_6	$(2.5, 2.7, 1.8, 0.5, 1.7, 0.8)$	0.0069	0.0607	-0.0493	0.0846	22
portfolio7_6_10	$(2.3, 2.4, 1.3, 1.4, 1.9, 0.9)$	0.0063	0.0814	-0.0464	0.0808	$5\frac{1}{2}$
portfolio8_6	$(0.0, 0.8, 6.4, 0.0, 0.8, 2.1)$	0.0120	0.1193	-0.1011	0.1668	20
portfolio8_6_10	$(0.6, 1.5, 4.7, 0.0, 1.8, 1.4)$	0.0102	0.1326	-0.0744	0.1290	$12\frac{1}{2}$
portfolio9_6	$(0.0, 2.0, 4.7, 0.0, 1.6, 1.6)$	0.0103	0.0956	-0.0753	0.1312	16
portfolio9_6_10	$(1.6, 2.1, 3.2, 0.0, 2.1, 1.1)$	0.0085	0.1054	-0.0563	0.1014	$5\frac{1}{2}$
portfolio10_6	$(2.3, 2.8, 2.2, 0.1, 1.7, 0.9)$	0.0075	0.0643	-0.0510	0.0888	19

Continued on next page

Problem	$\mathbf{10 * x}$	*P. Mean*	*P. Var*	*P. Skew*	*P. Kurt*	**Iters**
portfolio10_6_10	$(2.2, 2.4, 1.4, 1.3, 1.8, 0.9)$	0.0064	0.0825	-0.0467	0.0815	$5\frac{1}{2}$
portfolio11_6	$(1.9, 2.7, 2.6, 0.0, 1.7, 1.0)$	0.0080	0.0688	-0.0534	0.0942	19
portfolio11_6_10	$(2.1, 2.5, 1.6, 1.1, 1.8, 0.9)$	0.0067	0.0842	-0.0472	0.0828	$8\frac{1}{2}$
portfolio12_6	$(1.7, 2.5, 2.5, 0.0, 2.0, 1.2)$	0.0079	0.0687	-0.0510	0.0939	26
portfolio12_6_10	$(2.0, 2.4, 1.7, 1.0, 1.9, 1.0)$	0.0068	0.0857	-0.0469	0.0838	$10\frac{1}{2}$
portfolio13_6	$(3.0, 2.5, 0.5, 1.7, 1.6, 0.6)$	0.0053	0.0542	-0.0481	0.0788	21
portfolio13_6_10	$(2.5, 2.2, 0.8, 1.7, 1.9, 0.9)$	0.0057	0.0788	-0.0463	0.0793	$5\frac{1}{2}$
portfolio14_6	$(2.5, 2.6, 1.3, 1.0, 1.8, 0.8)$	0.0064	0.0577	-0.0472	0.0812	$21*$
portfolio14_6_10	$(2.3, 2.3, 1.1, 1.4, 1.9, 0.9)$	0.0061	0.0806	-0.0459	0.0803	$5\frac{1}{2}$
portfolio15_6	$(0.3, 1.7, 3.2, 0.0, 2.6, 2.2)$	0.0090	0.0827	-0.0551	0.1105	18
portfolio15_6_10	$(1.3, 1.8, 2.8, 0.0, 2.5, 1.6)$	0.0083	0.1045	-0.0510	0.0996	$11\frac{1}{2}$
portfolio16_6	$(2.6, 2.6, 1.2, 1.1, 1.7, 0.8)$	0.0062	0.0567	-0.0473	0.0803	21
portfolio16_6_10	$(2.3, 2.3, 1.1, 1.5, 1.9, 0.9)$	0.0060	0.0801	-0.0461	0.0800	$4\frac{1}{2}$
portfolio17_6	$(2.3, 2.7, 2.0, 0.3, 1.8, 0.9)$	0.0073	0.0630	-0.0498	0.0872	22
portfolio17_6_10	$(2.2, 2.4, 1.4, 1.3, 1.8, 0.9)$	0.0064	0.0823	-0.0465	0.0814	$6\frac{1}{2}$
portfolio18_6	$(1.6, 2.7, 3.0, 0.0, 1.7, 1.0)$	0.0083	0.0721	-0.0558	0.0986	20
portfolio18_6_10	$(2.0, 2.4, 1.7, 1.0, 1.9, 0.9)$	0.0068	0.0856	-0.0473	0.0838	$9\frac{1}{2}$
portfolio19_6	$(2.3, 2.7, 1.9, 0.3, 1.8, 0.9)$	0.0072	0.0625	-0.0493	0.0865	21
portfolio19_6_10	$(2.2, 2.4, 1.4, 1.3, 1.9, 0.9)$	0.0064	0.0822	-0.0464	0.0813	$5\frac{1}{2}$
portfolio20_6	$(2.4, 2.7, 1.7, 0.6, 1.8, 0.9)$	0.0068	0.0602	-0.0484	0.0839	21

Continued on next page

Problem	$10 * x$	*P. Mean*	*P. Var*	*P. Skew*	*P. Kurt*	Iters
portfolio20_6_10	$(2.2, 2.4, 1.3, 1.3, 1.9, 0.9)$	0.0063	0.0814	-0.0462	0.0808	$5\frac{1}{2}$
portfolio1_10	$(0.0, 3.4, 0.7, 4.4, 0.0, 0.0, 0.7, 0.6, 0.0, 0.2)$	0.0108	0.0887	-0.0739	0.1233	15
portfolio1_10_10	$(0.4, 2.1, 1.5, 3.1, 0.0, 0.8, 0.4, 0.9, 0.3, 0.5)$	0.0090	0.0994	-0.0528	0.0960	$4\frac{1}{2}$
portfolio2_10	–	–	–	–	–	–
portfolio2_10_10	$(0.0, 3.0, 1.1, 4.6, 0.0, 0.0, 0.3, 0.6, 0.0, 0.5)$	0.0108	0.1259	-0.0744	0.1235	$7\frac{1}{2}$
portfolio3_10	$(1.1, 2.4, 1.3, 1.9, 0.0, 0.0, 0.5, 1.4, 0.0, 1.4)$	0.0079	0.0577	-0.0435	0.0794	7
portfolio3_10_10	$(1.6, 1.2, 1.7, 1.1, 0.3, 1.0, 0.2, 1.4, 0.7, 0.9)$	0.0066	0.0734	-0.0380	0.0714	$2\frac{1}{2}$
portfolio4_10	$(0.6, 2.5, 1.1, 2.3, 0.0, 0.0, 0.6, 1.5, 0.0, 1.5)$	0.0084	0.0625	-0.0447	0.0852	$6\frac{1}{2}$
portfolio4_10_10	$(1.4, 1.4, 1.7, 1.4, 0.0, 1.0, 0.3, 1.4, 0.7, 0.8)$	0.0071	0.0766	-0.0380	0.0740	$2\frac{1}{2}$
portfolio5_10	$(0.5, 3.5, 1.3, 2.8, 0.0, 0.0, 0.4, 0.9, 0.0, 0.7)$	0.0092	0.0696	-0.0592	0.0993	$4\frac{1}{2}$
portfolio5_10_10	$(1.3, 1.7, 1.7, 1.7, 0.0, 0.9, 0.1, 1.3, 0.6, 0.7)$	0.0074	0.0791	-0.0410	0.0764	$2\frac{1}{2}$
portfolio6_10	$(2.2, 0.9, 1.7, 0.4, 0.5, 0.4, 0.4, 1.6, 0.3, 1.6)$	0.0057	0.0481	-0.0369	0.0669	17
portfolio6_10_10	$(1.8, 0.7, 1.7, 0.3, 1.0, 1.1, 0.1, 1.4, 0.8, 1.0)$	0.0055	0.0695	-0.0364	0.0681	$2\frac{1}{2}$
portfolio7_10	$(1.5, 2.4, 1.4, 1.6, 0.0, 0.0, 0.4, 1.5, 0.0, 1.4)$	0.0077	0.0562	-0.0441	0.0779	6
portfolio7_10_10	$(1.6, 1.1, 1.7, 0.9, 0.5, 1.0, 0.1, 1.4, 0.7, 0.9)$	0.0063	0.0720	-0.0383	0.0703	$2\frac{1}{2}$
portfolio8_10	$(0.0, 4.1, 0.0, 5.7, 0.0, 0.0, 0.2, 0.0, 0.0, 0.0)$	0.0121	0.1065	-0.0940	0.1500	16
portfolio8_10_10	$(0.0, 2.9, 1.1, 4.5, 0.0, 0.0, 0.4, 0.5, 0.0, 0.5)$	0.0107	0.1248	-0.0730	0.1221	$4\frac{1}{2}$
portfolio9_10	$(0.0, 3.4, 0.6, 4.4, 0.0, 0.0, 0.7, 0.6, 0.0, 0.3)$	0.0108	0.0887	-0.0738	0.1232	$3\frac{1}{2}$
portfolio9_10_10	$(0.5, 2.1, 1.5, 3.0, 0.0, 0.8, 0.4, 0.9, 0.4, 0.5)$	0.0090	0.0984	-0.0517	0.0949	$4\frac{1}{2}$
portfolio10_10	$(1.1, 2.7, 1.3, 1.9, 0.0, 0.0, 0.4, 1.3, 0.0, 1.3)$	0.0080	0.0586	-0.0466	0.0818	$6\frac{1}{2}$

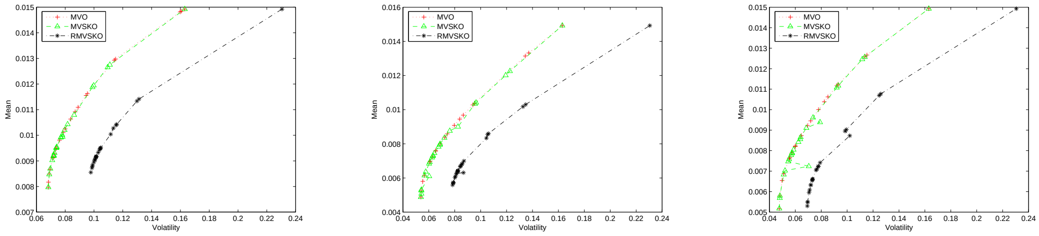
Continued on next page

Problem	$\mathbf{10 * x}$	*P. Mean*	*P. Var*	*P. Skew*	*P. Kurt*	**Iters**
portfolio10_10_10	(1.6, 1.3, 1.7, 1.1, 0.3, 1.0, 0.1, 1.4, 0.7, 0.9)	0.0066	0.0734	−0.0386	0.0715	$2\frac{1}{2}$
portfolio11_10	(0.6, 3.0, 1.0, 2.4, 0.0, 0.0, 0.5, 1.4, 0.0, 1.2)	0.0087	0.0642	−0.0500	0.0893	$5\frac{1}{2}$
portfolio11_10_10	(1.5, 1.5, 1.7, 1.4, 0.0, 1.0, 0.2, 1.3, 0.7, 0.8)	0.0071	0.0760	−0.0390	0.0736	$2\frac{1}{2}$
portfolio12_10	(0.4, 2.5, 1.0, 2.4, 0.0, 0.0, 0.6, 1.5, 0.0, 1.5)	0.0086	0.0641	−0.0449	0.0869	$5\frac{1}{2}$
portfolio12_10_10	(1.4, 1.4, 1.7, 1.5, 0.0, 1.0, 0.3, 1.4, 0.6, 0.7)	0.0072	0.0778	−0.0382	0.0750	$2\frac{1}{2}$
portfolio13_10	(2.2, 1.0, 1.7, 0.4, 0.4, 0.3, 0.3, 1.5, 0.2, 1.6)	0.0057	0.0473	−0.0372	0.0659	7
portfolio13_10_10	(1.8, 0.7, 1.7, 0.4, 1.0, 1.1, 0.1, 1.4, 0.8, 1.0)	0.0055	0.0695	−0.0370	0.0682	2
portfolio14_10	(1.7, 1.8, 1.5, 1.2, 0.0, 0.1, 0.5, 1.5, 0.0, 1.7)	0.0070	0.0520	−0.0389	0.0714	$6\frac{1}{2}$
portfolio14_10_10	(1.7, 0.9, 1.7, 0.7, 0.6, 1.1, 0.2, 1.4, 0.8, 0.9)	0.0061	0.0711	−0.0369	0.0694	$2\frac{1}{2}$
portfolio15_10	(0.0, 1.9, 0.0, 3.3, 0.0, 0.0, 1.0, 1.3, 1.2, 1.2)	0.0093	0.0782	−0.0432	0.1024	5
portfolio15_10_10	(0.0, 1.7, 0.8, 2.8, 0.0, 1.0, 0.6, 0.8, 1.4, 0.9)	0.0087	0.1020	−0.0408	0.0958	$7\frac{1}{2}$
portfolio16_10	(1.8, 1.8, 1.5, 1.0, 0.0, 0.2, 0.4, 1.5, 0.1, 1.6)	0.0068	0.0508	−0.0386	0.0699	7
portfolio16_10_10	(1.7, 0.9, 1.7, 0.7, 0.7, 1.1, 0.1, 1.4, 0.8, 1.0)	0.0060	0.0706	−0.0371	0.0691	$2\frac{1}{2}$
portfolio17_10	(1.2, 2.4, 1.3, 1.8, 0.0, 0.0, 0.5, 1.4, 0.0, 1.4)	0.0079	0.0577	−0.0438	0.0795	7
portfolio17_10_10	(1.6, 1.2, 1.7, 1.1, 0.3, 1.0, 0.2, 1.4, 0.7, 0.9)	0.0066	0.0732	−0.0381	0.0713	$2\frac{1}{2}$
portfolio18_10	(0.2, 3.2, 1.0, 2.7, 0.0, 0.0, 0.5, 1.3, 0.0, 1.1)	0.0091	0.0684	−0.0541	0.0956	$4\frac{1}{2}$
portfolio18_10_10	(1.4, 1.5, 1.7, 1.5, 0.0, 1.0, 0.2, 1.3, 0.6, 0.7)	0.0072	0.0777	−0.0397	0.0750	$2\frac{1}{2}$
portfolio19_10	(1.2, 2.3, 1.3, 1.8, 0.0, 0.0, 0.5, 1.4, 0.0, 1.5)	0.0077	0.0565	−0.0424	0.0777	7
portfolio19_10_10	(1.6, 1.2, 1.7, 1.0, 0.3, 1.0, 0.2, 1.4, 0.7, 0.9)	0.0065	0.0731	−0.0378	0.0711	$2\frac{1}{2}$
portfolio20_10	(1.4, 2.2, 1.3, 1.5, 0.0, 0.0, 0.4, 1.4, 0.0, 1.5)	0.0074	0.0539	−0.0409	0.0742	7

Continued on next page

Problem	$\mathbf{10 * x}$	***P. Mean***	***P. Var***	***P. Skew***	***P. Kurt***	**Iters**
portfolio20_10_10	(1.6, 1.1, 1.7, 0.9, 0.5, 1.0, 0.1, 1.4, 0.7, 0.9)	0.0063	0.0720	−0.0377	0.0702	$2\frac{1}{2}$
portfolio5_16	(0.0, 0.0, 2.4, 0.0, 0.0, 1.5, 0.0, 0.0, 1.7, 0.7, 0.0, 3.3, 0.0, 0.0, 0.0, 0.4)	0.0080	0.0623	−0.0498	0.0872	14
portfolio5_16_10	(0.8, 0.0, 1.7, 0.1, 0.4, 0.8, 0.5, 0.3, 1.0, 0.6, 0.1, 1.5, 0.6, 0.0, 0.5, 1.0)	0.0063	0.0683	−0.0374	0.0674	$4\frac{1}{2}$
portfolio10_16	(0.3, 0.0, 1.7, 0.0, 0.0, 1.0, 0.0, 0.0, 2.4, 0.6, 0.0, 2.1, 0.3, 0.0, 0.3, 1.2)	0.0069	0.0506	−0.0386	0.0702	14
portfolio10_16_10	(0.8, 0.0, 1.1, 0.5, 0.4, 0.6, 0.7, 0.6, 1.2, 0.4, 0.3, 1.0, 0.7, 0.1, 0.6, 1.0)	0.0054	0.0608	−0.0327	0.0606	$5\frac{1}{2}$
portfolio15_16	(0.0, 0.0, 1.5, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1.8, 0.6, 4.2, 0.0, 0.0, 0.0, 2.0)	0.0079	0.0703	−0.0439	0.0991	14
portfolio15_16_10	–	–	–	–	–	–
portfolio20_16	(0.8, 0.0, 1.3, 0.0, 0.0, 0.8, 0.0, 0.0, 2.5, 0.5, 0.0, 1.7, 0.5, 0.0, 0.6, 1.4)	0.0064	0.0468	−0.0345	0.0647	14
portfolio20_16_10	(0.8, 0.0, 0.9, 0.6, 0.3, 0.6, 0.7, 0.8, 1.3, 0.3, 0.3, 0.9, 0.6, 0.2, 0.6, 1.0)	0.0051	0.0590	−0.0313	0.0589	$5\frac{1}{2}$

Table 6: Partitioning procedure for POPs: portfolio weights and moments (assets from Dow 30)



(a) $n = 3$ (b) $n = 6$ (c) $n = 10$

Figure 2: MVO, MVSKO & RMVSKO Efficient Frontiers

6 Conclusions and Future Plans

The purpose of this paper was twofold. Firstly, we extended two convex finance models to their nonconvex analogues. In particular, we modelled the portfolio optimization problem and its worst-case, or robust, counterpart with higher order moments. To the best of our knowledge, it is the first work considering skewness and kurtosis in a robust framework with discrete uncertainty sets. Secondly, we handled the proposed models in a global optimization of polynomials framework using decomposition. The results obtained are certainly promising and support our belief that decomposition may play an important role in polynomial optimization and as a by-product in optimization in finance. However, several issues arise from this work and need to be taken into account. For example, the algorithm requires further investigation so as to be able to handle larger problems. What is more, the models addressed in this paper express the trade-off among the four portfolio moments through the objective function. Hence, they do not allow for skewness and/or kurtosis constraints. Such an amendment is essential should the investor require to enforce a lower bound on skewness and/or an upper bound on kurtosis. It is also essential for the generation of efficient surfaces, as opposed to efficient lines produced by the current models (see Figure 3).

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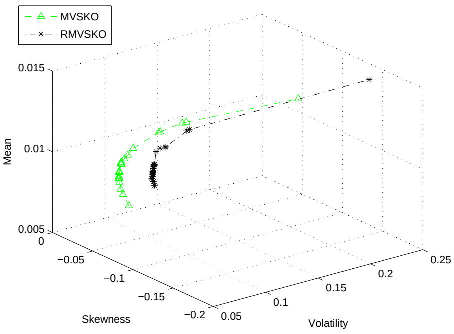
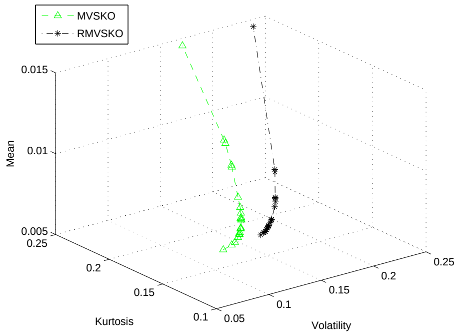
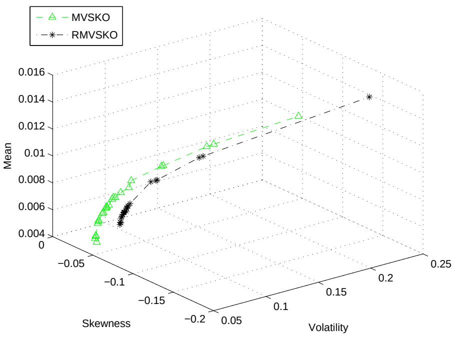


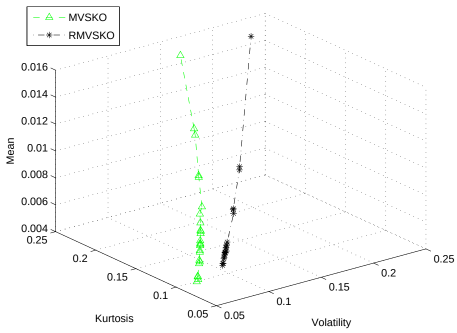
(a) $n = 3$: M-V-S



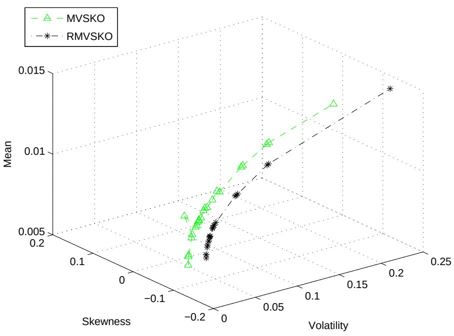
(b) $n = 3$: M-V-K



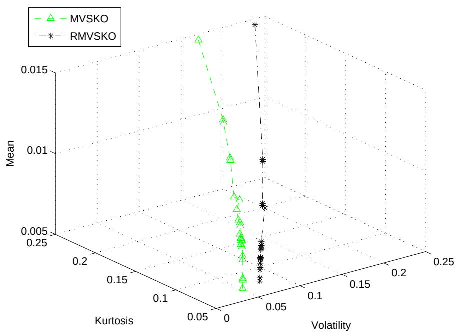
(c) $n = 6$: M-V-S



(d) $n = 6$: M-V-K



(e) $n = 10$: M-V-S



(f) $n = 10$: M-V-K

Figure 3: MVSKO & RMVSKO Efficient Lines

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