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Bootstrap Confidence Bands for Forecast Paths

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Bootstrap Confidence Bands for Forecast Paths

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Abstract

The problem of forecasting from vector autoregressive models has attracted considerable attention in the literature. The most popular non-Bayesian approaches use large sample normal theory or the bootstrap to evaluate the uncertainty associated with the forecast. The literature has concentrated on the problem of assessing the uncertainty of the prediction for a single period. This paper considers the problem of how to assess the uncertainty when the forecasts are done for a succession of periods. It describes and evaluates bootstrap method for constructing confidence bands for forecast paths. The bands are constructed from forecast paths obtained in bootstrap replications with an optimisation procedure used to find the envelope of the most concentrated paths. The method is shown to have good coverage properties in a Monte Carlo study.

1 Introduction

Vector autoregressive models (VARs) made popular by Sims (1980) are widely used in forecasting. While practitioners are usually concerned with forecasting the path that a variable (or a collection of variables) will follow, the literature has concentrated mostly on the problem of assessing the uncertainty associated with the prediction for a single period, either the next period or the one h-steps ahead. Two cases have received most attention: construction of asymptotic and bootstrap prediction intervals for single-variable forecast done for a particular period and construction of asymptotic and bootstrap confidence regions for multi-variable forecasts performed for a single period (see for example Kim (1999, 2001, 2004), Grigoletto (2005), Lütkepohl (2005)). In small samples the bootstrap methods of Efron (1979) have been shown to have better properties than asymptotic ones (Kim (1999, 2001), Grigoletto (2005)). More extensive evidence on the adequacy and advantages of the bootstrap has been obtained for the problem of constructing prediction intervals for forecasts calculated from AR model (see Masarotto (1990), Thombs and Schucany (1990), Kabaila (1993), Breidt at al. (1995), Clements and Taylor (2001), Kim (2002)).

This paper describes and evaluates the method for constructing a confidence band for a path of forecasts for a single variable in a stationary VAR. Jordà and Marcellino (2008) have recently discussed methods of band construction for paths based on asymptotic theory. The technique proposed in this paper uses the bootstrap and builds on the methods for constructing confidence bands for impulse response functions described in Staszewska (2007). The bands are constructed from forecast paths obtained in bootstrap replications.

Bootstrap samples are constructed from the backward representation of the VAR with the use of the bootstrap-after-bootstrap procedure of Kilian (1998). The bootstrap-after-bootstrap method has been previously employed for the construction of prediction regions by Kim (2001, 2004) and prediction intervals for AR models by Clements and Taylor (2001), Kim (2002) and Clements and Kim (2007). Using VAR's backward representation ensures that last observations in the pseudo-datasets are the same as in the original sample and guarantees that forecasts are conditional on past observations (see Thombs and Schucany (1990) in the context of AR forecasting and Kim (1999, 2001, 2004) and Grigoletto (2005) for VAR forecasting). The confidence band is then constructed as the envelope of the bootstrap forecast paths which are closest to the forecast path obtained on the basis of the original sample.

The coverage properties of confidence bands constructed according to the new method are investigated by means of Monte Carlo experiments. The performance of these bands is compared with the performance of the asymptotic Scheffé's and Bonferroni's methods described by Jordà and Marcellino (2008). As it is common in the literature to present the results of a forecasting exercise by showing a graph of the forecast path together with prediction intervals appropriate for making marginal inferences about single-period forecasts, coverage properties of one type of such naive band created by joining up bootstrap prediction intervals are also studied.

The main finding of the paper is that the bands created according to the new method of band construction have good coverage properties for both small and large sample sizes and for short and long forecasting horizons. In small samples their performance is better than the performance of the asymptotic confidence bands. The joined up prediction intervals perform very poorly and should not be used for making joint inferences about a succession of forecasts.

The outline of the paper is as follows. Section 2 considers the calculation of forecast path from a stationary VAR model. Sections 3 and 4 describe how the bootstrap samples are obtained and introduce the new method of constructing confidence band for the forecast path. The design and the results of the Monte Carlo experiments are given in Sections 5 and 6. Conclusions are presented in Section 7.

2 The forecast path

Consider the VAR(p) for K-dimensional vector of variables y:

$$y_t = A_0 + A_1 y_{t-1} + A_2 y_{t-2} + \ldots + A_p y_{t-p} + \varepsilon_t, \tag{1}$$

where A_0 is a $K \times 1$ vector of constants, A_i for $i = 1, \ldots, p$, are $K \times K$ coefficient matrices and ε_t is a $K \times 1$ vector of i.i.d. innovations. Innovations are such that $E(\varepsilon_t) = 0$ and $E(\varepsilon_t \varepsilon_t') = \Sigma_\varepsilon$, where Σ_ε is a positive definite matrix with finite elements. The model is stationary i.e. all roots of the characteristic equation $\det(I_K - A_1 z - \ldots - A_p z^p) = 0$ lie outside the unit circle.

In what follows, the forecast paths are obtained from (1) with the use of the iterated h-steps ahead predictor for the y_{T+h} vector, $h = 1, \ldots, H$, relying on the coefficients estimated from a sample of size T. For the asymptotic methods the h-steps ahead forecasts are calculated as:

$$\widehat{y}_T(h) = \widehat{A}_0 + \widehat{A}_1 \widehat{y}_T(h-1) + \widehat{A}_2 \widehat{y}_T(h-2) + \ldots + \widehat{A}_p \widehat{y}_T(h-p), \tag{2}$$

where $\widehat{y}_T(h-j) = y_{T+h-j}$ for $h-j \leq 0$ and $\widehat{A}_i$, for $i = 0, \ldots, p$, are the least squares estimates of the A_i parameters.

In the bootstrap method, to take into account the least squares parameter estimates bias, a bias-corrected predictor is considered (the particular form of the bootstrap bias-correction used is described in Section 3). In this case the forecasts are obtained from:

$$\widehat{y}^c_T(h) = \widehat{A}^c_0 + \widehat{A}^c_1 \widehat{y}^c_T(h-1) + \widehat{A}^c_2 \widehat{y}^c_T(h-2) + \ldots + \widehat{A}^c_p \widehat{y}^c_T(h-p), \tag{3}$$

where $\widehat{y}^c_T(h-j) = y_{T+h-j}$ for $h-j \leq 0$ and $\widehat{A}^c_i$, for $i = 0, \ldots, p$, are the bias-corrected estimates of the A_i parameters.

The forecast paths for $h = 1, \ldots, H$ for K variables are then sequences of values from $\widehat{y}_T(1)$, $\widehat{y}_T(2), \ldots, \widehat{y}_T(H)$ or $\widehat{y}^c_T(1), \widehat{y}^c_T(2), \ldots, \widehat{y}^c_T(H)$.

3 The bootstrap method

The proposed method of confidence band construction uses the residual (nonparametric) bootstrap to produce an indication of the uncertainty associated with the forecast path. Bootstrap samples are generated using the bootstrap-after-bootstrap procedure. The bootstrap DGP is based on the backward representation of the VAR ensuring that forecasts computed in the replications of the method are conditional on the last observations from the original sample.

The detailed steps of the bootstrap procedure are as follows:

a) using the available sample of data of size T, least squares is used to estimate the parameters of the forward VAR from (1) and the backward VAR of the form:

$$y_t = H_0 + H_1 y_{t+1} + H_2 y_{t+2} + \ldots + H_p y_{t+p} + \nu_t. \tag{4}$$

Residuals corresponding to parameter estimates $\widehat{A}_i$ and $\widehat{H}_i$, for $i = 0, \ldots, p$, denoted respectively by $\{\widehat{\varepsilon}_T\}$ and $\{\widehat{\nu}_T\}$, are obtained. The residuals are scaled as in Thombs and Schucany (1990),

b) in the loop with B_0 replications pseudo-datasets are generated from:

$$y^*_t = \widehat{H}_0 + \widehat{H}_1 y^*_{t+1} + \widehat{H}_2 y^*_{t+2} + \ldots + \widehat{H}_p y^*_{t+p} + \nu^*_t, \tag{5}$$

where the p initial values of y^* are set equal to the last p values of the original series and ν_t^* is a random draw with replacement from $\{\widehat{\nu}_T\}$, and also from:

$$y_t^* = \widehat{A}_0 + \widehat{A}_1 y_{t-1}^* + \widehat{A}_2 y_{t-2}^* + \ldots + \widehat{A}_p y_{t-p}^* + \varepsilon_t^*,$$

where the p initial values of the variables are set equal to the first p values of the original series and ε_t^* is a random draw with replacement from $\{\widehat{\varepsilon}_T\}$. The pseudo-datasets are used to estimate the parameters of, respectively, (4) and (1) yielding B_0 bootstrap estimates $\widehat{H}_i^*$ and $\widehat{A}_i^*$, for $i = 0, \ldots, p$. The parameter estimates biases are then calculated as: $bias(\widehat{H}_i) = \overline{\widehat{H}}_i^* - \widehat{H}_i$ and $bias(\widehat{A}_i) = \overline{\widehat{A}}_i^* - \widehat{A}_i$ and bias-corrected estimates are obtained according to: $\widehat{H}_i^c = \widehat{H}_i - bias(\widehat{H}_i)$ and $\widehat{A}_i^c = \widehat{A}_i - bias(\widehat{A}_i)$. In case such adjustment implies nonstationarity of the VAR, a stationarity correction is introduced as in Kilian (1998), resulting in slightly modified values of $\widehat{H}_i^c$ and $\widehat{A}_i^c$. Vectors of residuals $\{\widehat{\nu}_T^c\}$ and $\{\widehat{\varepsilon}_T^c\}$, corresponding to bias-corrected parameter estimates $\widehat{H}_i^c$ and $\widehat{A}_i^c$ and original series are calculated from, respectively, the backward and forward representations of the model,

c) B replications in the second loop of the bootstrap procedure are performed: pseudo-datasets are generated from the backward DGP with bias-corrected estimates:

$$y_t^{*c} = \widehat{H}_0^c + \widehat{H}_1^c y_{t+1}^{*c} + \widehat{H}_2^c y_{t+2}^{*c} + \ldots + \widehat{H}_p^c y_{t+p}^{*c} + \nu_t^{*c},$$

where the p initial values are set equal to the last p values of the original series and ν_t^{*c} is a random draw with replacement from $\{\widehat{\nu}_T^c\}$. On the basis of these pseudo-datasets the parameters of the forward model are estimated, resulting in parameter estimates $\widetilde{A}_i^*$. The estimates $\widetilde{A}_i^*$ are then corrected for the bias using the bias estimates obtained in the first bootstrap loop according to: $\widetilde{A}_i^{*c} = \widetilde{A}_i^* - bias(\widehat{A}_i)$, unless there is need for implementing the stationarity correction (see Kilian (1998)) resulting in adjusted values of $\widetilde{A}_i^{*c}$.

The bootstrap forecasts are then calculated as:

$$\widehat{y}_T^{*c}(h) = \widetilde{A}_0^{*c} + \widetilde{A}_1^{*c} \widehat{y}_T^{*c}(h-1) + \widetilde{A}_2^{*c} \widehat{y}_T^{*c}(h-2) + \ldots + \widetilde{A}_p^{*c} \widehat{y}_T^{*c}(h-p) + \varepsilon_{T+h}^{*c}, \tag{6}$$

for $h = 1, \ldots, H$, where $\widehat{y}_T^{*c}(h-j) = y_{T+h-j}$ for $h - j \leq 0$ and ε_{T+h}^{*c} is a random draw with replacement from $\{\widehat{\varepsilon}_T^c\}$.

The next section describes how bootstrap forecast values, calculated according to (6), can be used to construct confidence bands for forecast paths.

4 Methods of band construction

The $(1-\gamma)\times 100\%$ confidence band will be constructed by generating B bootstrap paths and discarding the $\gamma \times B$ most extreme paths. This method has been used for constructing confidence bands for impulse response functions by Staszewska (2007). As explained in that article, the idea can be implemented in different ways and the method used here, referred to as the closest paths method (CP), is a modification of that used previously. The method is implemented as follows (for the case of the k-th variable from the y vector, for $k = 1, \ldots, K$):

1. the forecast path is calculated from (3) and B bootstrap paths are obtained from (6),

2. for each period for which the paths are constructed (for $h = 1, \ldots, H$), the smallest and the largest bootstrap forecasts are identified and the paths they belong to are found. There are $2H$ such extreme values and maximally $2H$ paths they belong to,

3. for each of these bootstrap paths the distance from the forecast path obtained on the basis of the original sample is computed. The distance is treated in two ways, using either square or absolute errors. In the first case it is calculated as: $\sum_{h=1}^{H}(\widehat{y}^{c}_{k,T}(h) - \widehat{y}^{*c}_{k,T}(h))^2$, in the second according to: $\sum_{h=1}^{H}|\widehat{y}^{c}_{k,T}(h) - \widehat{y}^{*c}_{k,T}(h)|$,

4. a path which is furthest from the forecast path is found and removed from the set of B paths,

5. steps 2)-4) are repeated for $B-1$ remaining paths, then for $B-2$ and so on until $\gamma \times B$ paths have been removed,

6. the $(1-\gamma)\times 100\%$ confidence band is obtained as the envelope of the remaining $(1-\gamma)\times B$ bootstrap paths.

In the next section coverage properties of bands created according to this method are studied by means of Monte Carlo experiments. These coverage probabilities are compared with the coverage probabilities of alternative bands created according to one naive and two asymptotic methods of band construction.

Naive confidence bands are created from interval forecasts for single periods. A method of constructing bootstrap intervals considered here is the one proposed by Kilian (1998) in the context of impulse response analysis. The $(1-\gamma)\times 100\%$ prediction interval constructed for the k-th component of the y vector, for $k=1,\ldots,K$, at time $T+h$ is given by:

$$CI_{k,T}(h)=\left[s^*_{\gamma/2},s^*_{1-\gamma/2}\right],$$

where $s^*_{\gamma/2}$ and $s^*_{1-\gamma/2}$ are the $\gamma/2$ and $1-\gamma/2$ quantiles of the bootstrap distribution of $\widehat{y}^{*c}_{k,T}(h)$. The naive band is then created by joining up prediction intervals constructed for $h=1,\ldots,H$.

Asymptotic methods of band construction involve Scheffé's and Bonferroni's methods described by Jordà and Marcellino (2008). The Scheffé band for the k-th variable is constructed as:

$$\begin{bmatrix}\widehat{y}_{k,T}(1)\\ \widehat{y}_{k,T}(2)\\ \vdots\\ \widehat{y}_{k,T}(H)\end{bmatrix}\pm\begin{bmatrix}\sqrt{c^2_\gamma(1)}\\ \sqrt{\frac{c^2_\gamma(2)}{2}}\\ \vdots\\ \sqrt{\frac{c^2_\gamma(H)}{H}}\end{bmatrix}.\times P\mathbf{i}_H,$$

where $c^2_\gamma(h)$ is the critical value of a χ^2_h-distributed random variable at the $(1-\gamma)\times 100\%$ confidence level, P is a lower triangular matrix resulting from the Cholesky decomposition of the covariance matrix of the forecast path of the k-th variable $(\widehat{y}_{k,T}(1),\widehat{y}_{k,T}(2),\ldots,\widehat{y}_{k,T}(H))'$: $T^{-1}\widehat{\Xi}_{k,H}=PP'$ (the formula for $\widehat{\Xi}_{k,H}$ is given by Jordà and Marcellino (2008)) and $\mathbf{i}_H$ is an $H\times 1$ vector of ones.

The Bonferroni band for the k-th variable is obtained as:

$$\begin{bmatrix}\widehat{y}_{k,T}(1)\\ \widehat{y}_{k,T}(2)\\ \vdots\\ \widehat{y}_{k,T}(H)\end{bmatrix}\pm z_{\gamma/2H}\times diag(\widehat{\Xi}_{k,H})^{1/2},$$

where $z_{\gamma/2H}$ stands for the critical value of a standard normal random variable at an $\gamma/2H$ significance level and $diag(\widehat{\Xi}_{k,H})^{1/2}$ is an $H\times 1$ vector with the square roots of the diagonal entries of $\widehat{\Xi}_{k,H}$.

5 Experimental design

The design of the Monte Carlo experiments is to a large extend the same as in Jordà and Marcellino (2008). The DGP is based on the empirical VAR(4) with 3 variables and intercepts following the specification of Stock and Watson (2001). The variables include the rate of price inflation (P_t) (computed as the chain-weighted GDP price index), the unemployment rate (U_t) (measured by the civilian unemployment rate) and the federal funds rate (R_t). US quarterly data are used covering the period from 1960:I to 2004:I. The maximum likelihood parameter estimates provide parameter values in the Monte Carlo DGP. The DGP takes the form:

$$y_t = \begin{bmatrix} 1.076 \\ 0.125 \\ 0.347 \end{bmatrix} + \begin{bmatrix} 0.549 & -0.965 & 0.164 \\ 0.029 & 1.480 & 0.003 \\ 0.084 & -1.567 & 0.962 \end{bmatrix} y_{t-1} + \begin{bmatrix} 0.118 & 1.506 & -0.128 \\ -0.013 & -0.494 & 0.043 \\ 0.197 & 1.763 & -0.364 \end{bmatrix} y_{t-2} +$$
$$+ \begin{bmatrix} 0.060 & -0.954 & 0.054 \\ 0.002 & -0.029 & -0.024 \\ -0.070 & -0.848 & 0.333 \end{bmatrix} y_{t-3} + \begin{bmatrix} 0.261 & 0.250 & -0.098 \\ -0.012 & -0.014 & 0.008 \\ -0.046 & 0.563 & -0.010 \end{bmatrix} y_{t-4} + \varepsilon_t,$$

where $y_t = (P_t, U_t, R_t)'$ and the errors ε_t are assumed to be normally distributed with variance matrix: $\Sigma = \begin{bmatrix} 0.962 & -0.018 & 0.116 \\ -0.018 & 0.049 & -0.087 \\ 0.116 & -0.087 & 0.693 \end{bmatrix}$.

In the experiments a number of different sample sizes and forecasting horizons are considered. Samples of 60, 100 and 400 observations are analysed. The forecast horizons chosen are 1, 4, 8 and 12 periods. Bands for forecast paths of all three variables are considered. The confidence level $(1 - \gamma)$ is set to 0.68 or 0.95. Samples are generated from the DGP using random initial values of the variables. These are obtained by initializing the data generation for the first four values of all variables set to 0 and "observations" beginning from 100th used as the start-up values of the variables in the sample. The lag length of the VAR estimated in the Monte Carlo replications is determined, as in Jordà and Marcellino (2008), with the use of the AIC_C information criterion of Hurvich and Tsai (1993). The maximum lag length considered is 8.

The complete set-up of the Monte Carlo experiments, with 1000 replications, is the following:

- in each replication, a sample of data is generated from the Monte Carlo DGP. The DGP is run on beyond the sample size, to obtain the paths of true realisations of variables - the "true paths". The parameters of the VAR with automatically selected lag length are estimated on the basis of the "original sample". Asymptotic confidence bands are constructed. The bootstrap procedure is applied to construct the naive and closest paths confidence bands. The number of replications B is set to 2000 and B_0 is set to 1000. In the last step, it is checked whether the "true path" falls into the different confidence bands,

- the proportion of times that the various bands contain the "true path" is recorded and treated as the estimated coverage probabilities for the bands.

Before presenting the results of the Monte Carlo investigations it is useful to compare the bands created according to the different methods for a single sample. Figure 1 shows the forecast paths for P, U and R based on the Jordà and Marcellino (2008) dataset constructed for 8 periods ahead, together with 95% confidence bands involving joined up prediction intervals, Bonferroni's and Scheffé's bands and the closest paths squared band. The forecast path is represented by the solid line, dashed line indicates the joined up prediction intervals, the solid line with crosses corresponds to the Bonferroni band, solid line with boxes to the Scheffé band and solid line with circles to the band constructed according to the closest paths method.

The graph illustrates well quite different shapes of the bands. The most significant feature is that the closest paths band is the widest for initial periods while the Scheffé band lies outside the remaining bands for later periods. The joined up confidence intervals are generally the narrowest apart from period 1 for which the band can be wider than the Scheffé band.

Of course it is possible to provide bands corresponding to different confidence levels and plot them on a single diagram. The result would be like the fan charts used by the Bank of England to present uncertainty about future levels of inflation; see e.g. Bank of England (2008). The Bank's charts, however, seem to be constructed by joining up confidence intervals for forecasts done for single periods.

6 Results

The coverage probabilities estimated for different confidence bands in the Monte Carlo experiments are given in Tables 1-2 of the Appendix. Tables 3-4 report additionally the average widths of the bands measured as sum of bands' spreads at periods covered by the forecasting horizon.

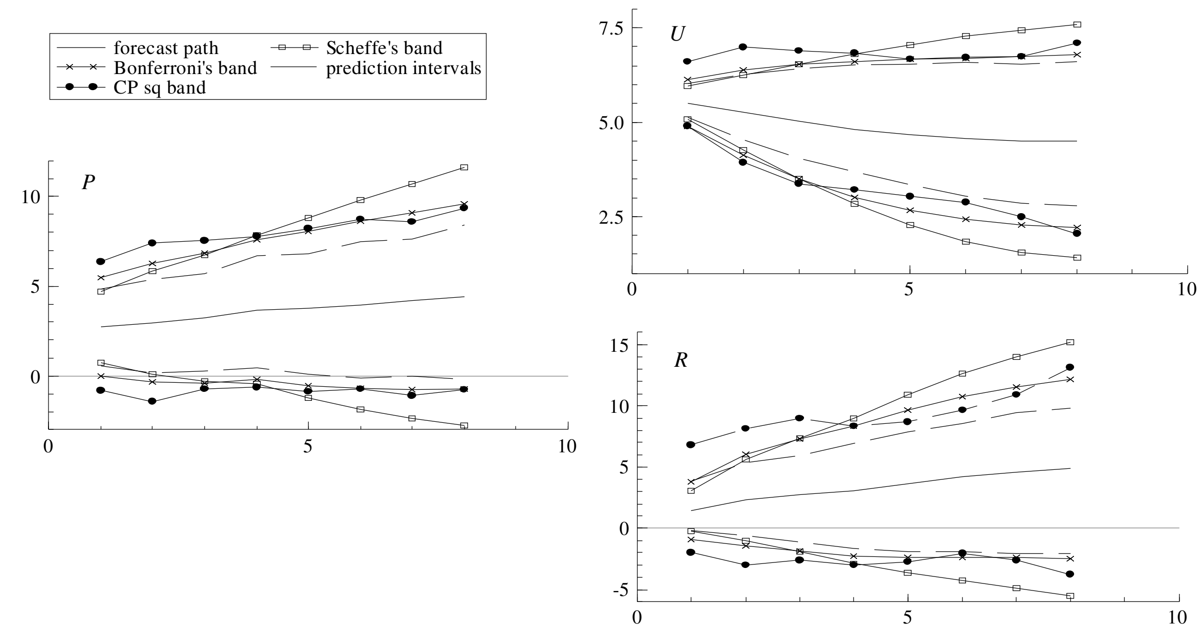

Figure 1: 8-steps ahead forecast paths for the Jordà and Marcellino (2008) dataset together with different 95% confidence bands

It can be seen that the naive method of joining up a number of prediction intervals works rather badly. For forecasting horizons beyond 1 the naive bands have coverage probabilities for the paths substantially below the designed confidence levels. The probabilities decrease as the forecasting horizon is extended. The bands are the narrowest, but given their poor properties it cannot be considered as an advantage.

The performance of the asymptotic bands is much better, especially for the largest sample size and confidence level of 0.95. In smaller samples, for longer forecasting horizons the Bonferroni bands have coverage probabilities which exceed the nominal confidence level, while the Scheffé bands tend to underachieve these values. The bands are much wider than the naive ones. For longer forecasting horizons the Scheffé bands open out and become wider than the Bonferroni ones.

Of all the methods, the closest paths method performs the best. Its properties are good for both smaller and larger sample sizes, shorter and longer forecasting horizons and both confidence levels. The two versions of the method, in which distance is interpreted in an absolute and squared error sense have very similar properties. The bands are wider than the naive ones. For shorter forecasting horizons they can be also wider than both types of asymptotic bands, but for longer horizons they become narrower than the Scheffé bands. They may be wider or narrower than the

Bonferroni bands depending on the confidence level which influences the coverage properties of the latter.

7 Conclusions

This paper has described and evaluated a new method for constructing confidence bands for forecast paths obtained from a VAR. The evaluation was by means of Monte Carlo experiments using a DGP and sample sizes of the kind used in macroeconomics. The method was shown to achieve accurate coverage probabilities for different sample sizes, forecasting horizons and confidence levels. Its performance was compared with that of the asymptotic methods recently proposed by Jordà and Marcellino (2008) and it performed better, especially in small samples.

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Appendix

Table 1. Estimated coverage probabilities of different 68% confidence bands

	$T = 60$			$T = 100$			$T = 400$		
	P	U	R	P	U	R	P	U	R
$H = 1$									
pred. interv.	0.656	0.662	0.651	0.660	0.671	0.668	0.659	0.693	0.703
Bonf.	0.619	0.630	0.616	0.634	0.645	0.645	0.653	0.685	0.696
Schef.	0.619	0.630	0.616	0.634	0.645	0.645	0.653	0.685	0.696
CP abs	0.664	0.658	0.649	0.650	0.673	0.673	0.662	0.690	0.704
CP sq	0.664	0.658	0.649	0.650	0.673	0.673	0.662	0.690	0.704
$H = 4$									
pred. interv.	0.322	0.431	0.379	0.299	0.421	0.359	0.297	0.466	0.360
Bonf.	0.658	0.726	0.684	0.696	0.763	0.714	0.742	0.824	0.802
Schef.	0.489	0.570	0.523	0.502	0.580	0.521	0.537	0.649	0.605
CP abs	0.681	0.638	0.659	0.666	0.657	0.645	0.665	0.699	0.691
CP sq	0.686	0.651	0.666	0.674	0.660	0.650	0.662	0.695	0.678
$H = 8$									
pred. interv.	0.215	0.292	0.258	0.213	0.276	0.262	0.144	0.267	0.245
Bonf.	0.675	0.755	0.703	0.788	0.774	0.746	0.803	0.840	0.813
Schef.	0.463	0.554	0.514	0.534	0.572	0.547	0.532	0.633	0.597
CP abs	0.675	0.659	0.658	0.721	0.660	0.665	0.676	0.665	0.675
CP sq	0.674	0.666	0.656	0.728	0.668	0.668	0.677	0.670	0.675
$H = 12$									
pred. interv.	0.167	0.242	0.210	0.164	0.189	0.201	0.099	0.200	0.187
Bonf.	0.674	0.757	0.712	0.768	0.789	0.760	0.811	0.862	0.850
Schef.	0.461	0.575	0.499	0.516	0.558	0.546	0.524	0.631	0.645
CP abs	0.703	0.689	0.658	0.742	0.684	0.697	0.688	0.705	0.695
CP sq	0.699	0.702	0.658	0.738	0.688	0.698	0.695	0.703	0.696

Table 2. Estimated coverage probabilities of different 95% confidence bands

	$T = 60$			$T = 100$			$T = 400$		
	P	U	R	P	U	R	P	U	R
$H = 1$									
pred. interv.	0.944	0.947	0.946	0.948	0.940	0.938	0.949	0.954	0.95
Bonf.	0.921	0.925	0.928	0.926	0.928	0.917	0.95	0.952	0.948
Schef.	0.921	0.925	0.928	0.926	0.928	0.917	0.95	0.952	0.948
CP abs	0.948	0.946	0.944	0.950	0.945	0.938	0.952	0.953	0.949
CP sq	0.948	0.946	0.944	0.950	0.945	0.938	0.952	0.953	0.949
$H = 4$									
pred. interv.	0.843	0.841	0.848	0.839	0.868	0.841	0.847	0.895	0.886
Bonf.	0.903	0.904	0.896	0.923	0.926	0.914	0.955	0.954	0.953
Schef.	0.846	0.873	0.870	0.878	0.893	0.885	0.913	0.940	0.938
CP abs	0.941	0.907	0.925	0.941	0.933	0.919	0.952	0.941	0.948
CP sq	0.941	0.906	0.922	0.947	0.936	0.924	0.949	0.939	0.946
$H = 8$									
pred. interv.	0.784	0.828	0.796	0.817	0.822	0.796	0.794	0.826	0.798
Bonf.	0.871	0.914	0.871	0.923	0.935	0.923	0.949	0.969	0.968
Schef.	0.818	0.903	0.867	0.899	0.917	0.893	0.921	0.943	0.929
CP abs	0.920	0.907	0.918	0.946	0.937	0.934	0.939	0.940	0.951
CP sq	0.921	0.906	0.917	0.950	0.941	0.935	0.937	0.939	0.947
$H = 12$									
pred. interv.	0.772	0.775	0.771	0.803	0.801	0.807	0.794	0.826	0.798
Bonf.	0.857	0.921	0.884	0.928	0.928	0.931	0.949	0.969	0.968
Schef.	0.814	0.898	0.864	0.884	0.907	0.902	0.921	0.943	0.929
CP abs	0.920	0.932	0.937	0.952	0.947	0.946	0.939	0.940	0.951
CP sq	0.920	0.939	0.934	0.952	0.947	0.943	0.937	0.939	0.947

Table 3. Average width of different 68% confidence bands

	$T = 60$			$T = 100$			$T = 400$		
	P	U	R	P	U	R	P	U	R
				$H = 1$					
pred. interv.	2.201	0.480	1.871	2.114	0.465	1.788	1.977	0.447	1.684
Bonf.	2.047	0.448	1.743	2.000	0.439	1.687	1.947	0.439	1.655
Schef.	2.047	0.448	1.743	2.000	0.439	1.687	1.947	0.439	1.655
CP abs	2.204	0.480	1.872	2.110	0.464	1.785	1.970	0.446	1.678
CP sq	2.204	0.480	1.872	2.110	0.464	1.785	1.970	0.446	1.678
				$H = 4$					
pred. interv.	11.916	4.024	13.013	11.081	3.876	12.448	9.752	3.589	11.401
Bonf.	18.679	6.291	20.384	17.729	6.179	19.895	16.741	6.144	19.542
Schef.	17.569	6.302	20.229	16.564	6.207	19.739	15.306	6.181	19.385
CP abs	18.430	5.621	19.023	17.338	5.415	18.277	15.579	5.033	16.849
CP sq	18.409	5.625	19.145	17.292	5.427	18.387	15.465	5.042	16.932
				$H = 8$					
pred. interv.	30.875	11.687	36.656	27.740	10.874	34.120	23.375	9.751	30.545
Bonf.	53.761	20.406	63.765	50.127	19.594	61.237	46.575	19.382	60.661
Schef.	58.106	22.929	71.870	54.370	21.892	69.308	49.800	21.716	69.144
CP abs	52.413	18.353	59.045	48.020	17.215	55.426	42.059	15.653	50.351
CP sq	52.326	18.286	58.916	47.853	17.125	55.328	41.744	15.579	50.181
				$H = 12$					
pred. interv.	53.943	21.449	66.314	48.720	19.621	62.455	39.385	17.002	52.791
Bonf.	96.967	39.213	118.130	91.067	37.430	115.650	83.914	36.323	111.970
Schef.	114.600	45.509	143.420	109.110	41.974	141.890	100.200	40.316	138.170
CP abs	94.705	36.060	111.750	87.442	33.488	106.200	74.081	29.707	92.474
CP sq	94.396	35.928	111.400	87.188	33.317	105.790	73.657	29.535	91.964

Table 4. Average width of different 95% confidence bands

	$T = 60$			$T = 100$			$T = 400$		
	P	U	R	P	U	R	P	U	R
$H = 1$									
pred. interv.	4.315	0.948	3.680	4.175	0.914	3.515	3.910	0.878	3.320
Bonf.	4.034	0.884	3.435	3.941	0.866	3.325	3.838	0.865	3.261
Schef.	4.034	0.884	3.435	3.941	0.866	3.325	3.838	0.865	3.261
CP abs	4.311	0.944	3.680	4.160	0.910	3.501	3.890	0.874	3.302
CP sq	4.311	0.944	3.680	4.160	0.910	3.501	3.890	0.874	3.302
$H = 4$									
pred. interv.	23.628	7.985	25.863	21.911	7.679	24.645	19.262	7.088	22.498
Bonf.	26.649	8.976	29.082	25.294	8.816	28.384	23.885	8.766	27.881
Schef.	27.215	9.580	30.953	25.696	9.435	30.206	23.833	9.398	29.681
CP abs	28.841	9.348	30.907	26.975	9.021	29.606	24.127	8.396	27.298
CP sq	28.868	9.291	30.854	26.967	8.957	29.530	23.983	8.336	27.193
$H = 8$									
pred. interv.	62.116	23.544	73.975	55.259	21.726	68.188	46.216	19.306	60.440
Bonf.	71.577	27.169	84.896	66.740	26.088	81.532	62.010	25.806	80.764
Schef.	81.017	31.638	99.452	75.816	30.250	95.925	69.583	30.022	95.690
CP abs	79.940	29.546	93.527	71.701	27.343	86.368	61.487	24.558	77.220
CP sq	79.823	29.396	93.236	71.630	27.194	86.129	61.353	24.374	76.973
$H = 12$									
pred. interv.	110.590	43.641	136.160	97.819	39.373	125.830	77.894	33.671	104.690
Bonf.	125.360	50.693	152.720	117.730	48.388	149.510	108.480	46.957	144.750
Schef.	151.880	59.987	189.280	144.530	55.508	187.210	132.830	53.401	182.370
CP abs	145.610	56.964	177.010	130.060	51.657	163.920	106.250	44.516	138.470
CP sq	145.090	56.666	176.070	129.750	51.341	163.020	105.940	44.215	137.620