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Index Mutual Fund Replication

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Abstract

This paper discusses the application of an index tracking technique to mutual fund replication problems. By using a tracking error (TE) minimization method and two tactical rebalancing strategies (i.e. the calendar based strategy and the tolerance triggered strategy), a multi-period fund tracking model is developed that replicates S&P 500 mutual fund returns. The impact of excess returns and loss aversion on overall tracking performance is also discussed in two extended cases of the original TE optimization respectively. An evolutionary method, namely Differential Evolution, is used for optimizing the asset weights. According to the experiment results, it is found that the proposed model replicates the first two moments of the fund returns by using only five equities. The TE optimization strategy under loss aversion with tolerance triggered rebalancing dominates other combinations studied with regard to tracking ability and cost efficiency.

Key words. Passive Portfolio Management, Fund Tracking, Multi-Period Optimization, Differential Evolution.

1 Introduction

In the last decade, individual holdings of corporate stocks have decreased while holdings through fund management institutions have correspondingly increased. According to the Investment Company Institute's official survey, the combined assets of U.S. mutual funds reached a peak of 12 trillion dollars in May 2008; although there was a great redemption pressure on the fund industry due to the recent credit crunch, the net asset value of the funds was in excess of 9 trillion dollars at the end of 2008. As the survey shows, approximately half of the fund holdings were claimed and managed by equity funds. The latter typically choose

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from three management styles, namely active management, passive management, or a blend of the two. The literature shows that most of actively managed equity funds underperform their passive benchmark portfolios after adjustments are made for fund management fees and expenses. For example, actively managed funds usually do not outperform index mutual funds which are passively managed to mimic certain indices in the long-term (see Malkiel [1995], Carhart [1997], Bogle [1999] and Haslem et al. [2008]). According to the official survey, the number of U.S. index funds increased almost three-fold from 134 to 373, in which the number of S&P 500 index funds rose from 72 to 124 at a growth rate of over 70% in the last decade. And more recently, the institutional investment in index funds was increased dramatically after the bankruptcy of major investment banks on Wall Street in September 2008.

While investing in funds brings several advantages (such as professional management) over direct investments in equities, expenses such as management fees or distribution fees have continued to increase during the last two decades. The question of whether charging higher fund fees benefits investors has been discussed at great length (see Anderson and Ahmed [2005]). Furthermore, researchers have drawn attention to a confusing phenomenon in the fund market: while fund fees and expenses vary quite a lot, the return patterns of the funds typically show relatively small amounts of dispersion. Therefore, it is not necessary to use expensive funds if the performance of funds are similar. As Sharpe [1966] has pointed out, funds with lower expenses tend to have higher reward-to-risk ratios; and investors should avoid using expensive funds since the high fund fees reduce the overall payout.

This paper extends index tracking techniques to perform index fund return replication. From the literature, several index tracking methods have been discussed by researchers: the classic tracking error (TE) minimization method of Roll [1992]; the principal components factor model and the cointegration based index tracking method in Alexander and Dimitriu [2004, 2005]. The classic index TE minimization model has been widely studied by researchers. Meade and Salkin [1990] used quadratic programming to construct equity index funds by minimizing tracking errors between index returns and asset returns which were generated from the autoregressive conditional heteroskedastic (ARCH) process. Montfort et al. [2008] applied an optimized sampling method in order to select equities to minimize tracking errors. As an alternative to traditional numerical methods, heuristic methods provide ways of approaching difficult combinatorial optimization problems, especially of solving financial optimization problems. An interesting feature of heuristic methods is that they combine stochastic search with supervised search; this can provide investors with new and efficient methods to obtain good solutions for complex and constrained optimization problems. Traditional deterministic optimization approaches tend to be poorly served for these problems whose solution space becomes rough and discontinuous after imposing different types of constraints on optimization problems. Pioneers in computer science and finance have employed heuristic methods to tackle index tracking problems. For example, Beasley et al. [2003] proposed an evolutionary heuristic method to obtain portfolio compositions which tracked market indices. Gilli and Këllezi [2002], and Maringer [2006a, 2008] adopted Threshold Accepting and Differential Evolution, respectively, to solve constrained index tracking problems. A comprehensive survey of heuristic method applications in index tracking can be found in di Tollo and Maringer [2009].

There are three main reasons to study the fund replication problem by using index tracking methods. First, although index mutual funds are usually considered to be cheap, they tend to outperform most of the actively managed funds in the long-term according to the literature. Secondly, the application of index tracking techniques in index mutual fund replications has not been widely discussed in the literature yet; this paper therefore discusses such applications in order to address this issue. Thirdly, if the index fund returns can be replicated, the dispersion in respect of the fund fees and expenses should not be great. Several important issues which are not usually included in the index tracking discussion are addressed in this study. For example, the proposed multi-period tracking model involves the tactical rebalancing issue, the transaction cost limitation and the cash reservation issue. A cardinality constraint (i.e. using a subset of market equities to track funds) is also imposed to the optimization problem, as fund managers are reluctant to disclose their actual holdings.

The paper is organized as follows: Section 2 introduces the fund tracking model and a population based heuristic method for tackling the multi-period optimization problem; Section 3 provides the results and discussions from the in-sample and out-of-sample experiments; Section 4 draws conclusions.

2 Tracking Error Minimization and Multi-period Readjustment

The optimal solution for the fund tracking problem is a portfolio composed of a subset of equities. If a fund adopts a passive management, its holding composition should not change dramatically. In other words, if the tracker successfully replicates the fund over the in-sample period, the behaviour of the tracker and its target should be similar over certain out-of-sample periods. The study proposes to perform the fund replication in two stages: a construction stage and an adjustment stage. This section starts by introducing the two-stage optimization model, and then presents a heuristic method for solving the optimization problem.

2.1 The Optimization Problem for Tracker Construction

At the construction stage, an investor observes the daily prices of N equities, as well as the daily net asset values (NAV) of target funds over historical time $[T_\varpi, T_0]$. Let T_ϖ and T_0 denote the first day and the end of the construction stage respectively. At time T_0, an optimal set of k equities ($k < N$) and holding quantities n_{i,T_0} need to be known in order to construct a tracker which best replicates its target over the period $[T_\varpi, T_0]$. To measure the similarity of tracker returns and fund returns, a difference measure between the returns is adopted, i.e. the tracking error $TE = \sqrt{\frac{1}{T}\sum_t (r_{P,t} - r_{I,t})^2}$. The market value of the tracker at time t is the sum of the market value of holdings, $P_t = \sum_{i=1}^{N} n_{i,t} \cdot S_{i,t}$. Since the short selling of equities is not considered in this study, $n_{i,t}$ must be positive integers. Let $S_{i,t}$ denote the market price of equity i at time t, and the tracker daily return at time t is defined as $r_{P,t} = \ln(P_t/P_{t-1})$. The NAV reflects the dollar value of one share of a fund, which is used to compute the fund return at time t: $r_{I,t} = \ln(NAV_t/NAV_{t-1})$.

The tracker portfolio is constructed with three constraints. Suppose that an initial budget B_{T_0} is available at T_0, and an amount of $P_{T_0} = \sum_{i=1}^{N} n_{i,T_0} \cdot S_{i,T_0} \leq B_{T_0} \cdot (1 - C)$ is used to purchase a set of non-negative and integer quantities n_{i,T_0} of equities from the market, where C is the initial cash reserve rate. If an equity has a price S_{i,T_0} at time T_0, the weight invested can be written as $x_{i,T_0} = (n_{i,T_0} \cdot S_{i,T_0})/P_{T_0}$. For the purpose of portfolio diversification, if an equity i is included, its corresponding weight should satisfy two weight constraints, $x_g^\ell \leq x_{i,T_0} \leq x_g^u$, otherwise its holding quantity n_{i,T_0} should be zero. After taking out the value P_{T_0} from the initial budget B_{T_0}, one has the initial cash reserve: $Cash_{T_0} = B_{T_0} - P_{T_0}$, which is used to cover transaction costs. Secondly, the tracker can only use a subset k out of the market equities N to track funds. Therefore, a cardinality constraint $\sharp\mathcal{C}_g = \sum_{i=1}^{N} I_{\mathcal{C}_g}(i) \leq k$ is imposed to control the number of equities purchased. $\mathcal{C}_g$ is the equity set; k the number of equities purchased; and $I_{\mathcal{C}_g}(i)$ is an indicator function. It should be noted that introducing lower and upper limits also incurs implicit cardinality constraints: if the weight of each equity must be kept below a weight x_g^u, at least $k^{\min} = \lfloor 1/x_g^u \rfloor$ equities must be bought; on the other hand, the maximum number of equities which should be purchased for satisfying the minimum weight constraint is $k^{\max} = \lceil 1/x_g^\ell \rceil$. Finally, the tracker has an upper limit on the costs of each transaction: the costs cannot grow beyond the amount which is a small proportion γ of the tracker value P_t, i.e. $TC_t \leq \gamma \cdot P_t$. The transaction costs TC are set as linear functions of the amount for buying the equities. Thus, the costs can be modelled as $TC_{T_0} = \sum_{i \in \mathcal{C}_g} \rho \cdot n_{i,T_0} \cdot S_{i,T_0}$, where ρ is defined as a transaction cost coefficient. Although the current work uses linear transaction cost functions, the model can handle the transaction cost functions when the solution spaces are not continuous and convex.

For ease of reading, the TE optimization problem is summarized using the following notations.

$$\min_{\mathbf{n}} TE = \sqrt{\frac{\sum_t (r_{P,t} - r_{I,t})^2}{T_0 - T_\varpi}}$$

s.t.

$$\begin{gathered}
n_{i,T_0} \in \mathbb{N}_0^+ \\
t \in [T_\varpi, T_0] \\
k^{\min} < \sharp\mathcal{C}_g = \sum_{i=1}^{N} I_{\mathcal{C}_g}(i) \leq k < k^{\max} \\
x_g^\ell \leq \frac{n_{i,T_0} \cdot S_{i,T_0}}{P_{T_0}} \leq x_g^u \quad \text{for } i \in \mathcal{C}_g \\
TC_{T_0} = \sum_{i \in \mathcal{C}_g} \rho \cdot n_{i,T_0} \cdot S_{i,T_0} \leq \gamma \cdot P_{T_0}
\end{gathered}$$

n_{i,T_0}	number of shares of the i-th equity invested at time T_0
γ	the transaction cost limiting ratio
P_t	market value of the tracker at time t
$r_{P,t}$	tracker return at time t
$r_{I,t}$	index fund return at time t
$\mathcal{C}_g$	tracker equity set
x_g^ℓ	minimum weight of each equity
x_g^u	maximum weight of each equity
TC_t	transaction cost at time t
ρ	transaction cost coefficient
$Cash_t$	cash reserve at time t
C	cash reserve rate
B_t	sum of the tracker market value and cash reserve at time t
$S_{i,t}$	per-share market value of the i-th equity at time t
N	number of available equities in the equity market

While the market moves over time, the tracker holdings should be revised if the tracker return drifts away from its target return. The following subsection introduces the rebalancing problem and two rebalancing strategies.

2.2 Tracker Rebalancing Stage

At rebalancing time T_j, an optimal adjustment set $\delta(n_{i,T_j})$ of k equities should be known. After rebalancing, the replicator should best track its target over the period $[T_{j-1}, T_j]$. The decision variables are a set of adjusted quantities $\delta(n_{i,t})$, which can be either positive or negative integers. The holding of equity i after the rebalancing at time $t = T_j$ can be written as: $n_{i,T_j} = n_{i,T_{j-1}} + \delta(n_{i,T_j})$. As the rebalancing involves both selling and buying, the transaction cost is modeled as twice the cost at the construction stage: $TC_{T_j} = \sum_{i \in \mathcal{C}_g} 2 \cdot \rho \cdot |n_{i,T_j} - n_{i,T_{j-1}}| \cdot S_{i,T_j}$. If the cash reserve is not enough to cover transaction costs, the model will recover the cash reserve by liquidating assets, which depends on the reserve rate C, the tracker value P_t, and the optimal holdings of the next period. At the rebalancing stage, all the constraints from the construction stage must be satisfied. Thus the optimization problem at this stage is summarized as follows.

$$\min_{\delta(\mathbf{n})} TE = \sqrt{\frac{\sum_t (r_{P,t} - r_{I,t})^2}{T_j - T_{j-1}}}$$

s.t.

$$\begin{gathered}
\delta(n_{i,T_j}) \in \mathbb{Z} \\
n_{i,T_j} \in \mathbb{N}_0^+ \\
t \in [T_{j-1}, T_j] \\
k^{\min} < \sharp\mathcal{C}_g = \sum_{i=1}^{N} I_{\mathcal{C}_g}(i) \leq k < k^{\max} \\
x_g^\ell \leq \frac{(n_{i,T_{j-1}} + \delta(n_{i,T_j})) \cdot S_{i,T_j}}{P_{T_j}} \leq x_g^u \quad \text{for } i \in \mathcal{C}_g \\
TC_{T_j} = \sum_{i \in \mathcal{C}_g} 2 \cdot \rho \cdot |n_{i,T_j} - n_{i,T_{j-1}}| \cdot S_{i,T_j} \leq \gamma \cdot P_{T_j}
\end{gathered}$$

2.3 Rebalancing Strategies

Two portfolio rebalancing strategies are usually adopted by market practitioners (see Eakins and Stansell [2007]). One is portfolio readjustment at regular calendar interval (e.g. quarterly), which is referred to as calendar based rebalancing. The other is a tolerance triggered strategy which is based on waiting until triggers reach certain thresholds.

2.3.1 Calendar Based Rebalancing

This strategy schedules regular rebalancing at a regular calendar interval T_ψ.

1. The model splits the future time horizon $[T_0, T_\omega]$ into M subintervals $[T_0, T_1]$, $[T_1, T_2]$, $\cdots$, $[T_{M-1}, T_\omega]$ according to a fixed calendar interval T_ψ. The interval number M is decided by the length of the rebalancing stage and the time interval: $M = \lfloor (T_\omega - T_0)/T_\psi \rfloor$.

2. At the rebalancing time T_j, the model

   (a) decides an optimal set of quantities $\delta\left(n_{i,T_j}\right)$ based on the market information over the time period $[T_{j-1}, T_j]$,

   (b) adjusts portfolio holdings $n_{i,T_j} = n_{i,T_{j-1}} + \delta(n_{i,T_j})$,

   (c) updates cash reserves $Cash_{T_j} = Cash_{T_{j-1}} - TC_{T_j}$, and

   (d) waits till the next planned rebalancing point $T_{j+1} = T_j + T_\psi$.

3. The model repeats the second step until the end of the rebalancing stage T_ω.

2.3.2 Tolerance Triggered Rebalancing

The second strategy employs a rolling window that starts moving from an historical time. There are M check-points T_j in the rebalancing stage $[T_0, T_\omega]$, following $T_j = j \cdot \wp$, $j = 1, 2, ..., M$ and $M = \lfloor (T_\omega - T_0)/\wp \rfloor$. $\wp$ is the rolling step. At each check-point, the model computes trigger values based on the tracker performance over the current window which starts at an historical time $T_{\varsigma,j} = T_j - W_L$, where W_L is the window length. Three values are considered as trigger tolerances for the current problem: the TE tolerance ξ_1; the lower equity weight limit x_g^ℓ; and the upper equity weight limit x_g^u. The rebalancing procedure is described as follows.

1. At each check-point T_j, the tracker has the starting point of the j-th window, $T_{\varsigma,j} = T_j - W_L$ with $j = 1, 2, ..., M$.

2. (a) If any one of the following conditions is violated: $\sqrt{\frac{1}{W_L} \cdot \sum_{t=T_{\varsigma,j}}^{T_j} |r_{P,t} - r_{I,t}|^2} < \xi_1$, $\frac{n_{i,T_j} \cdot S_{i,T_j}}{P_{T_j}} > x_g^\ell$, and $\frac{n_{i,T_j} \cdot S_{i,T_j}}{P_{T_j}} < x_g^u$, the model
   (i) finds an optimal set of $\delta(n_{i,T_j})$ based on the market information in the time period $[T_{\varsigma,j}, T_j]$,
   (ii) adjusts portfolio holdings: $n_{i,T_j} = n_{i,T_{j-1}} + \delta(n_{i,T_j})$,
   (iii) updates cash reserves: $Cash_{T_j} = Cash_{T_{j-1}} - TC_{T_j}$;

   (b) otherwise the model keeps the holdings unchanged: $n_{i,T_j} = n_{i,T_{j-1}}$;

(c) the model waits till the next check-point $T_{j+1} = T_j + \wp$.

3. The model repeats the second step up till the end of rebalancing stage T_ω.

2.4 Extensions of Traditional Tracking Error Optimization

2.4.1 Extension to Include Excess Return

Gilli and Këllezi [2002] considered the following average positive deviations from market benchmarks, or the average excess return (ER):

$$ER = \tfrac{1}{T_N}\textstyle\sum_t (r_{P,t} - r_{I,t}), \qquad \text{for } r_{P,t} \geq r_{I,t}$$

as a part of the index tracking objective, where T_N represents the number of returns observed over the period. The model considers index fund return as the benchmark, and modifies the classic TE optimization objective as follows:

$$\min \lambda \cdot TE - (1 - \lambda) \cdot ER.$$

λ is a value between 0 and 1, representing the weighting difference of the measure between TE and ER.

2.4.2 Extension to Include Loss Aversion

The classic TE minimization objective cannot distinguish between positive and negative deviations of the tracker relative to its target, due to ignorance of the sign of return deviations. Loss averse investors tend to strongly prefer avoiding losses to acquiring gains, therefore the behaviour can be modelled by introducing an aversion coefficient ϑ to the TE measure with $\vartheta > 1$ (see Maringer [2008]).

$$\widetilde{\Delta_r} = \begin{cases} r_{P,t} - r_{I,t} & r_{P,t} \geq r_{I,t} \\ (r_{P,t} - r_{I,t}) \cdot \vartheta & r_{P,t} < r_{I,t} \end{cases}.$$

Thus the original objective function at the construction and rebalancing stage is modified to:

$$\min \widetilde{TE} = \sqrt{\frac{1}{T_N}\sum_t (\widetilde{\Delta_r})^2}.$$

2.5 The Optimization Method

Heuristic methods provide ways of tackling combinatorial optimization problems, and they are most suitable for solving constrained financial optimization problems. Differential Evolution (DE) is a population based heuristic method which was originally proposed by Storn and Price [1997] for the optimization problems with continuous space. It generates new solutions by combining three solutions, and cross-over with a fourth solution. After the linear combination and cross-over, the generated solution would replace the current best solution if the generated solution has a higher fitness value. More specifically, for each current solution $\boldsymbol{v_p}$, a new solution $\boldsymbol{v_c}$ is generated from the following procedure: randomly selecting three different members from the current population ($p_1 \neq p_2 \neq p_3 \neq p$); and then linearly combining the solution vectors at probability π_1, or inheriting the original p-th solution at probability otherwise. With the standard DE method, only the population size P, the scaling factor F and the cross-over probability π_1 need to be considered. A variant of DE takes the advantage of diversity which is brought from having noise in the linear combination and cross-over, to escape from local optima convergence and avoid premature convergence. The noise is generated by adding normally distributed random numbers to F value and the difference of two solution vectors respectively. Vectors $\boldsymbol{z_1}$ and $\boldsymbol{z_2}$ represent the extra noise in the algorithm; they contain random numbers being zero with probability $\boldsymbol{\pi_2}$ and $\boldsymbol{\pi_3}$ respectively, or following normally independent distribution $N(0, \sigma_1^2)$ and $N(0, \sigma_2^2)$. The linear combination and cross-over procedure are described as follows:

$$v_c[i] := \begin{cases} v_{p1}[i] + (F + z_1[i]) \cdot (v_{p2}[i] - v_{p3}[i] + z_2[i]) & \text{with probability } \pi_1 \\ v_p[i] & \text{otherwise,} \end{cases}$$

where π_1 is the cross-over probability. After the linear combination and cross-over, DE asses the fitness of the solution, which is defined as the negative objective function values, due to the minimization problem in the model. More specifically, if the fitness value of $\boldsymbol{v_c}$ is higher than the one of $\boldsymbol{v_p}$, the solution $\boldsymbol{v_p}$ is replaced by $\boldsymbol{v_c}$, and the updated $\boldsymbol{v_p}$ exists in the current population; otherwise $\boldsymbol{v_p}$ survives. The process is repeated until a halting criterion is met. The DE algorithm is described by using the pseudo code as follows.

Since the solutions from DE may be either positive or negative, the no-short-selling constraint would be violated if one directly took $\boldsymbol{v_p}$ as weight solutions. In other words, the solutions from DE may not be valid for the current problem. According to the literature, one may use penalty functions to impair the fitness of solutions which violate constraints. Despite the straightforwardness of using penalty functions, computational efficiency would be reduced if one intended to use penalty functions to satisfy all constraints. A mapping function is used to translate $\boldsymbol{v_p}$ into valid equity weights, in order to satisfy the integer, cardinality and weight constraints. The mapping function first checks the number of positive elements κ in $\boldsymbol{v_p}$. If $\kappa \leq 0$, the function prohibits the $\boldsymbol{v_p}$ entering the fitness evaluation procedure. Otherwise, the mapping function selects the k largest positive elements in $\boldsymbol{v_p}$ giving $\kappa > k$. If the positive number satisfies the condition $\kappa < k^{\min}$, the function picks $k^{\min}$ largest elements from $\boldsymbol{v_p}$. If the positive number of elements

Algorithm 1 Differential Evolution.

randomly initialize population of vectors $\boldsymbol{v_p}$, p= 1...P
while the halting criterion is not met **do**
  for all current solutions $\boldsymbol{v_p}$, p=1...P **do**
    randomly pick $p_1 \neq p_2 \neq p_3 \neq p$
    $v_c[i] \leftarrow v_{p1}[i]+(F+z_1[i])(v_{p2}[i]-v_{p3}[i]+z_2[i])$ with probability π_1, or $v_c[i] \leftarrow v_p[i]$ otherwise
    interpret $\boldsymbol{v_c}$ into equity weights
    compute the fitness value of v_c
  end for
  for the current solution $\boldsymbol{v_p}$, $p = 1...P$ **do**
    if Fitness($\boldsymbol{v_c}$) > Fitness($\boldsymbol{v_p}$) **then** $\boldsymbol{v_p} \leftarrow \boldsymbol{v_c}$
  end for
end while

satisfies the condition $k^{\min} < \kappa < k$, those equities with positive values are included in the tracker. For further details of the mapping function please refer to Maringer and Oyewumi [2007].

The included equities are first assigned the minimum weight x_g^ℓ, and then they are increased in proportion to the values in $\boldsymbol{v_p}$ until the sum of them add up to unity. If an equity weight exceeds the maximum weight, its weight is decreased to x_g^u, and the excess part is superadded proportionally to other selected equities according to their weights. The optimal holding of the i-th equity is computed by rounding up $x_{i,t}P_t/S_{i,t}$ to the closest integer. Usually, a simple rounding approximation may bring neutrality bias to the final solutions of the problems with binary variables, e.g. the problem for yes/no decisions. However, an-unit difference in stock holdings due to the rounding approximation could be usually ignored when the holding amount of a stock is large. The mapping function has been employed in previous research; and it is found that the function works properly for similar problems (see Maringer and Oyewumi [2007]). In addition to the mapping function, a penalty function is used to guarantee the solutions satisfying the transaction cost constraint. The penalty function impairs the fitness value of the unsatisfied solutions by imposing a punishment: $-\max(CT_t - 2 \cdot \gamma \cdot P_t, 0)$.

2.6 Experiment Settings and Data

The following experiment settings were used: Initial budget $B_t = 10,000,000$ dollars; Cash Reserve Rate $C = 10\%$; Transaction Cost Limiting Rate $\gamma = 1\%$; Transaction Cost Coefficient $\rho = 0.1\%$; Cardinality Size $k = 5$. Maringer and Oyewumi [2007] have shown that increasing the cardinality size helps to reduce the TE. However, this study aims at constructing 'small' trackers by using limited assets. To the portfolios with fewer dimensions, the transaction costs will be cheap; and the costs can be easily modelled if the cost functions are nonlinear. Furthermore, a well diversified portfolio can be achieved by using limited assets (see Maringer [2006b]). Therefore, the cardinality size in this study was set at a relative low value, say 5. The calendar based rebalancing strategy considered a rebalancing time interval T_ψ being 60 trading days, representing quarterly readjustment. In the tolerance triggered rebalancing, the window size W_L was set at 60. As a result, the trigger values were computed by using the information in the last 60 trading days. A step size $\wp = 10$ was adopted, i.e. the minimum rebalancing frequency was two weeks to prevent the rebalance from occurring too frequently. The TE value was used as a trigger in the tolerance triggered rebalancing. The TE tolerance ξ_1 was set at 0.004, which is proposed to set the value as twice the in-sample tracking errors. The equity weight trigger $x_{i,t}$ had minimum and maximum values being 1% and 50%, respectively. Setting such a high upper limit helps to study the actual diversification ability of the tracking model, as the DE algorithm has greater degrees of freedom in choosing asset weights. To the real-world applications, the upper limit should be reduced further to avoid the risk due to sudden changes of major index shares (in this study it was found that the maximum weights usually varied around 40%). In the TE and ER optimization experiments, the weighting difference λ was set at 50%. In the TE with loss aversion experiments, the aversion coefficient ϑ was set at 2, thereby doubling the impairment of negative deviations.

The technical parameters of DE algorithm are listed as follows. Population size and iteration number were set at 1,000 and 4,000; the factor F was set at a value 0.5; and the cross-over probability π_1 was at 60%. The parameters were used to generate the artificial noise: $\pi_2 = 50\%$, $\pi_3 = 10\%$, $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.1$. The above settings was found to be highly suited for solving this index mutual fund tracking problem; in preliminary experiments, the relative differences between the TE and its corresponding lowest TE recorded after independent restarts were found to be zero or small.

A total of 445 equities were used to track index mutual funds. The following five S&P 500 index funds were considered as targets in this study: ETRADE S&P 500 Index; Vanguard 500 Index; USAA S&P 500 Index; UBS S&P 500 Index A; and TIAA-CREF S&P 500 Index Retire. The five index funds were traced by Tracker 1 to 5, respectively. The data comprise of daily prices and the net asset values of the equities and funds in the period January 2004 to December 2007, downloaded from Datastream. The equities have price sequences with 1,043 observations. To decide whether a large or a small data sample is suitable for this fund replication, different in-sample data sizes are considered at the construction stage and at the rebalancing stage. The first 250 observations (i.e. the information in 2004) were used to construct trackers, which would be held from the beginning of 2005. At the rebalancing stage, the latest 60 observations at each rebalancing point were employed to decide on the optimal adjusted quantities.

3 Experiment Results

3.1 In-Sample Tracking Performance

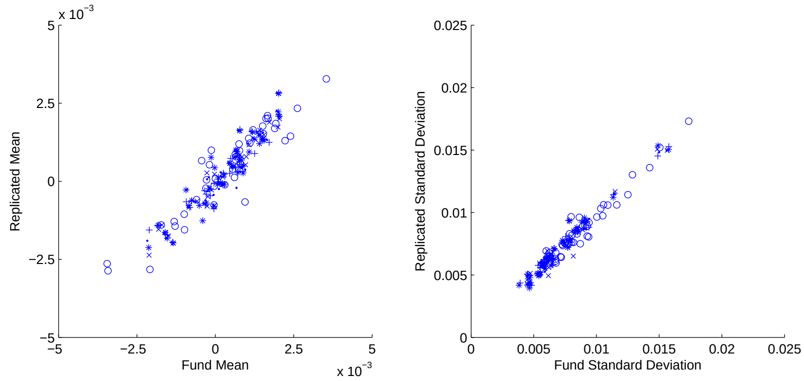
In order to determine how well the model replicates the fund returns over the in-sample periods, Figure 1 shows the scatter plot of the means and standard deviations of the actual and replicated daily returns at monthly intervals by using the calendar based rebalancing. The first to fifth replicated case are identified by using different markers: Plus sign, Circle, Asterisk, Point and Cross. It can be seen that there are linear relationships existing between the fund return moments and replicated return moments. In order to show clearly the impact of the three objective functions on the in-sample tracker performance, Figure 2 provides the TE and the excess return to risk ratio (or the excess Sharpe ratio $(r_p/\sigma_p - r_I/\sigma_I)$), which were computed at monthly intervals. The return to risk ratio was approximately equal to the Sharpe Ratio when one considers the daily safe rate being tiny. In the figure, different line styles (i.e. solid, solid-circle, dashed, dotted, and dash-dot) tell the performance of the five trackers respectively. In Figure 2, the left and right panels show the in-sample TE and the excess Sharpe ratio respectively. As the figure shows, the excess Sharpe ratios are improved after considering the ER and loss aversion; and the TE from the two extended objective functions are higher than that from the classic TE optimization as expected. For the tolerance triggered rebalancing, the in-sample tracker performance from the three objective functions should have been similar to those from the calendar based rebalancing, since the in-sample results are independent of the rebalancing strategies.

The next subsection provides the results from the out-of-sample experiments and analysis in order to judge whether the model replicates the fund returns over the out-of-sample periods. The impact of the objective functions and the two rebalancing strategies on tracker performance are further explored and discussed.

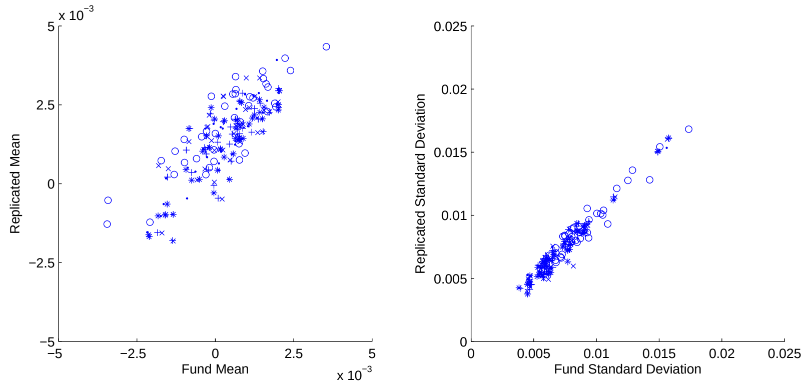
3.2 The Out-of-Sample Analysis of Replicators

Figure 3 and Figure 4 provide scatter plots of the return means and return standard deviations. The scatters were computed on the basis of the replicated returns and fund returns at monthly intervals.

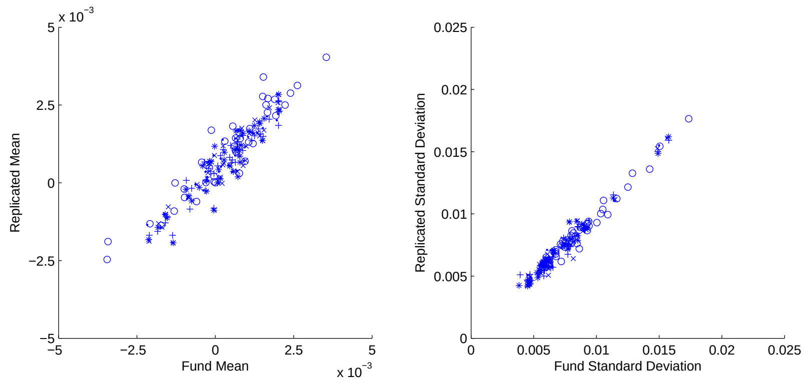
In order to quantify the relationship between the fund return and tracker return moments, linear regression analysis is employed to study the observations shown in the scatter plots. Table 2 and Table 3 provide the regression analysis



(a) Tracking Error Optimization

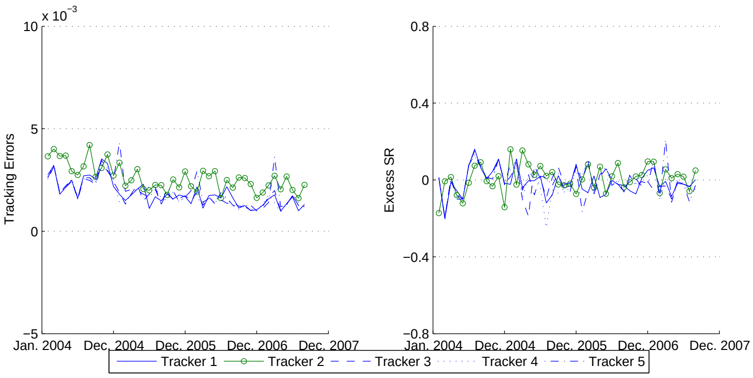


(b) Tracking Error with Excess Return Optimization

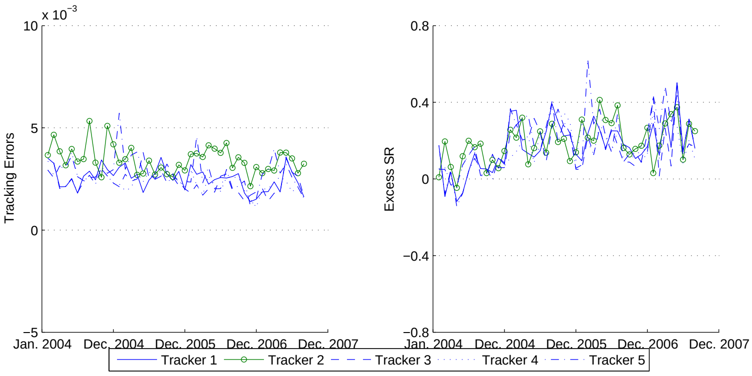


(c) Tracking Error with Loss Aversion Optimization

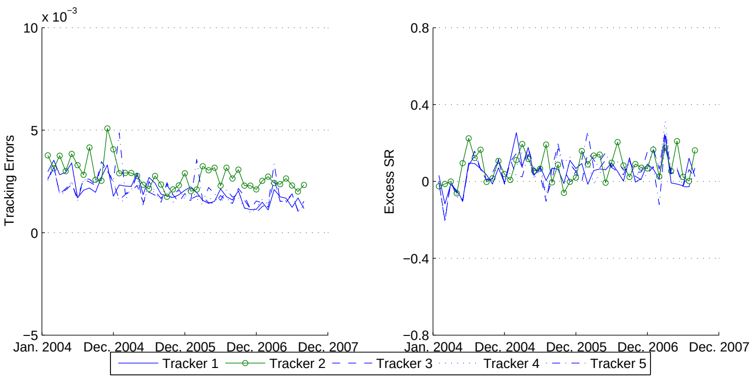
Figure 1: In-Sample Monthly Means and Standard Deviations of Actual and Replicated Returns from Calendar Based Rebalancing.



(a) Tracking Error Optimization



(b) Tracking Error and Excess Return Optimization



(c) Tracking Error and Loss Aversion Optimization

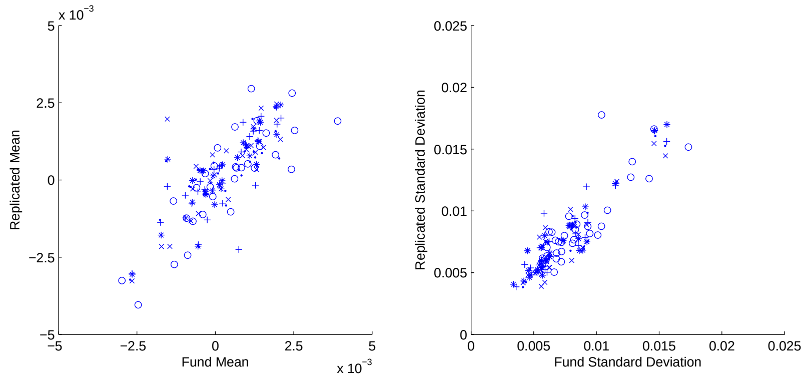
Figure 2: In-Sample Monthly Tracking Errors and Excess Sharpe Ratios from Calendar Based Rebalancing.

results. As the two tables show, the R^2 values from the standard deviation regression are all higher than the values from the mean regression, indicating that the model replicates the return standard deviations better than the return means. The intercept α and slope β values from the analysis can be used as indicators to evaluate tracking performance: if a tracker perfectly replicates a fund, the regression intercept and slope should be equal to 0 and 1 respectively. According to the table, most of the intercept α values from the mean regression are close to 0, and not statistically different from 0. Although the α values in the standard deviation regression case are small, they are statistically different from 0 at a 5% confidence level, indicating the risk of trackers are slightly higher than their targets. The impact of the extra risk on tracker performance will be discussed together with the excess Sharpe ratio.

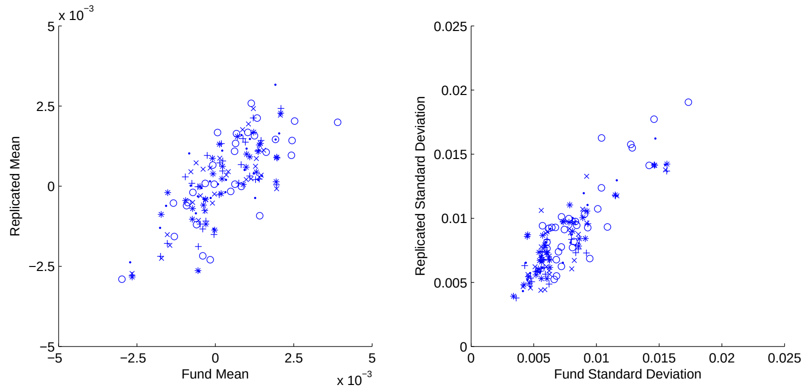
When using the calendar based rebalancing, most of the β values are lower than 1, and statistically different from 1. However, while using the tolerance triggered rebalancing, there is the evidence that the β values are not statistically different from 1. Therefore, the tracker using the tolerance triggered strategy should have had a higher tracking ability than the one using the calendar based strategy. Moreover, the β values from the two extended objective functions are higher than the one from the classic TE optimization when the trackers employ the tolerance triggered strategy. Due to the strict constraints imposed, it is reasonable that the replication criteria, i.e. the $\alpha = 0$ and $\beta = 1$ criteria are not perfectly satisfied. However, the current tracking performance could be improved further by relaxing some constraints, e.g. increasing the cardinality size.

It may be interesting to compare further the tracker performance of the model using the rebalancing strategies. The sub-figures in Figure 5 report the TE computed at monthly intervals; and the left and right panels of Figure 5 show the TE as a result of using the calendar based rebalancing and tolerance triggered rebalancing respectively. There is no significant difference found between the TEs which are optimized by using the two different rebalancing strategies. In other words, setting the TE tolerance ξ_1 as twice the in-sample TE achieved the same result as that from the calendar based strategy case.

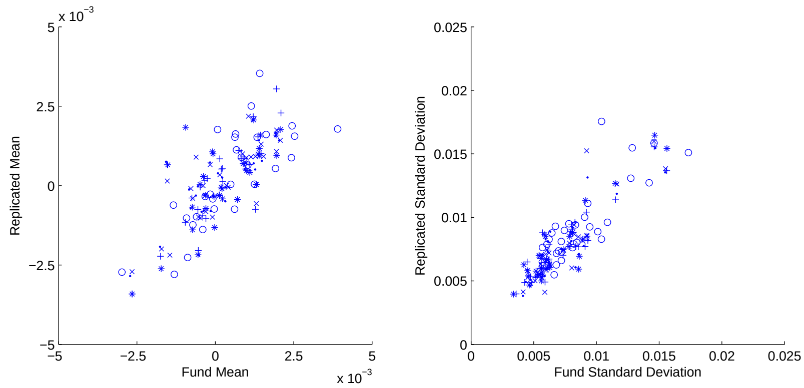
Figure 6 shows the impact of different objective functions and rebalancing strategies on the excess Sharpe ratio over the out-of-sample periods. The upper panel of Figure 6 reveals that there are consistent negative deviations of the ratios in year 2005. While considering the ER maximization as a part of the objective function, the negative excess Sharpe ratios in the year are reduced. However, it should be noted that both the positive and negative deviation of the ratio are larger than those in the upper panel. In the lower panel, the effect of loss aversion can be seen. The negative deviations of the excess Sharpe ratios are reduced, while the positive deviations can still be maintained at the same magnitude as those in



(a) Tracking Error Optimization

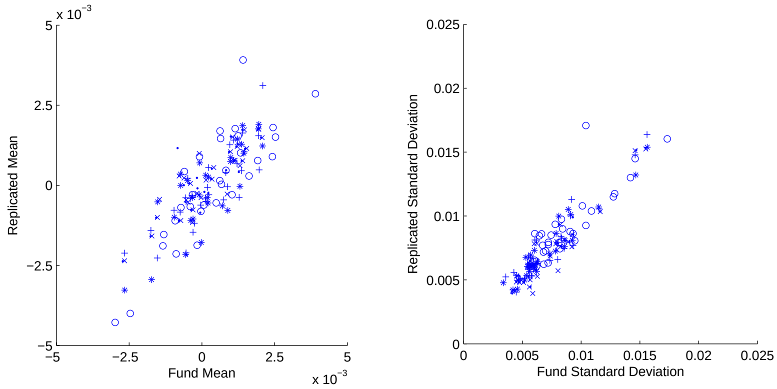


(b) Tracking Error with Excess Return Optimization

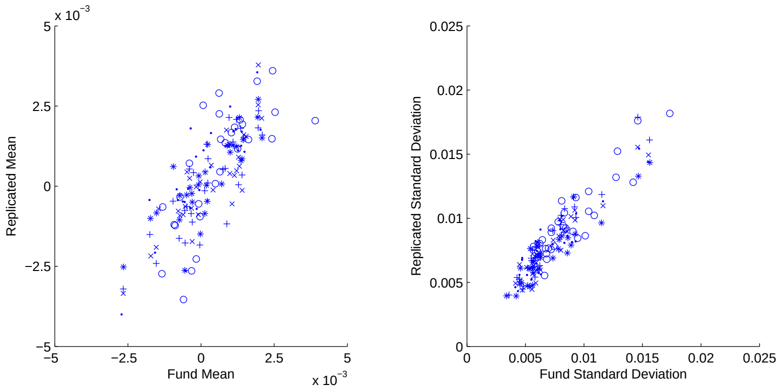


(c) Tracking Error with Loss Aversion Optimization

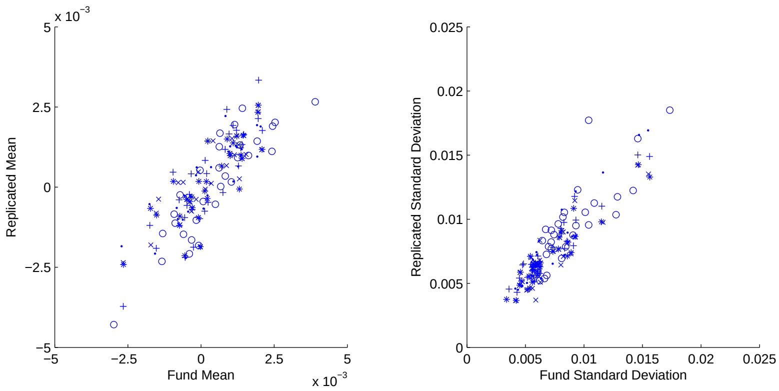
Figure 3: Out-of-Sample Means and Standard Deviations of Actual and Replicated Returns from Calendar Based Rebalancing.



(a) Tracking Error Optimization



(b) Tracking Error with Excess Return Optimization



(c) Tracking Error with Loss Aversion Optimization

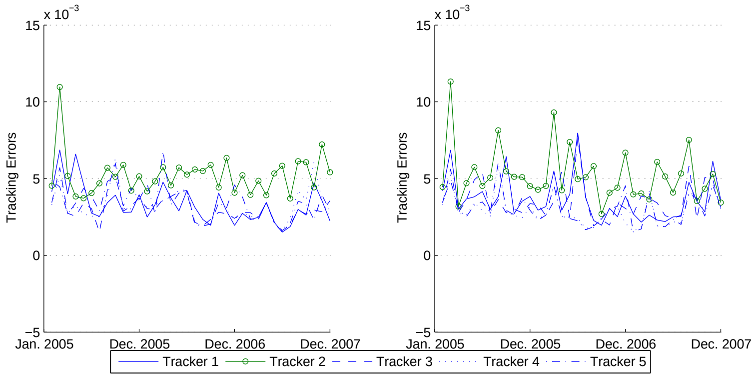
Figure 4: Out-of-Sample Means and Standard Deviations of Actual and Replicated Returns from Tolerance Triggered Rebalancing.

Table 2: Regression Analysis of Actual and Replicated Return Means.

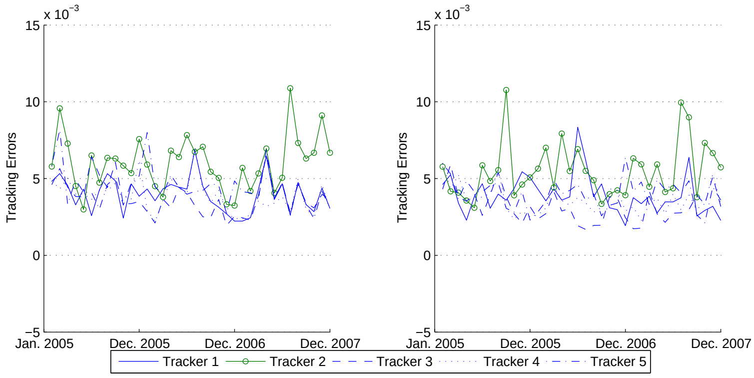
	TE Opt.		Ext. 1		Ext. 2	
	C.	T.	C.	T.	C.	T.
SSE.(10^{-3})	0.0983	0.0818	0.1296	0.1427	0.1110	0.0919
R^2	0.6494	0.6924	0.5576	0.6682	0.6117	0.7094
$\bar{\alpha}$ (10^{-3})	-0.0199	-0.1942	-0.1054	-0.0692	-0.0969	-0.1239
std.(α) (10^{-3})	0.0661	0.0603	0.0758	0.0796	0.0702	0.0639
$p(\alpha \neq 0)$	0.7436	0.0016	0.1665	0.3860	0.1696	0.0542
$\bar{\beta}$	0.8950	0.8999	0.8474	1.1238	0.8771	0.9931
std.(β)	0.0532	0.0485	0.0610	0.0640	0.0565	0.0514
$p(\beta \neq 1)$	0.0497	0.0413	0.0124	0.0536	0.0308	0.8966

Table 3: Regression Analysis of Actual and Replicated Return Standard Deviations.

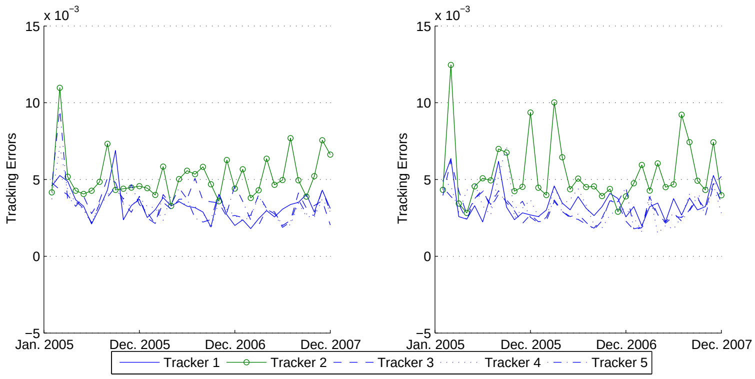
	TE Opt.		Ext. 1		Ext. 2	
	C.	T.	C.	T.	C.	T.
SSE(10^{-3})	0.2516	0.1623	0.3170	0.1736	0.2793	0.2316
R^2	0.8206	0.8650	0.7597	0.8664	0.7908	0.8294
$\bar{\alpha}$(10^{-3})	0.5485	0.7363	1.6700	1.1030	1.0100	0.6546
std.(α)(10^{-3})	0.2920	0.2346	0.3278	0.2426	0.3077	0.2802
$p(\alpha \neq 0)$	0.0623	0.0020	$< .0001$	$< .0001$	0.0013	0.0208
$\bar{\beta}$	0.9654	0.9176	0.9008	0.9548	0.9245	0.9548
std.(β)	0.0365	0.0293	0.0410	0.0303	0.0384	0.0350
$p(\beta \neq 1)$	0.3472	0.0051	0.0163	0.1362	0.0495	0.1973



(a) Tracking Error Optimization



(b) Tracking Error with Excess Return Optimization



(c) Tracking Error with Loss Aversion Optimization

Figure 5: Out-of-Sample tracking errors from using the three objective functions and two rebalancing strategies (left – calendar based rebalancing, right – tolerance triggered rebalancing).

Table 4: Rebalancing Times at Rebalancing Stage.

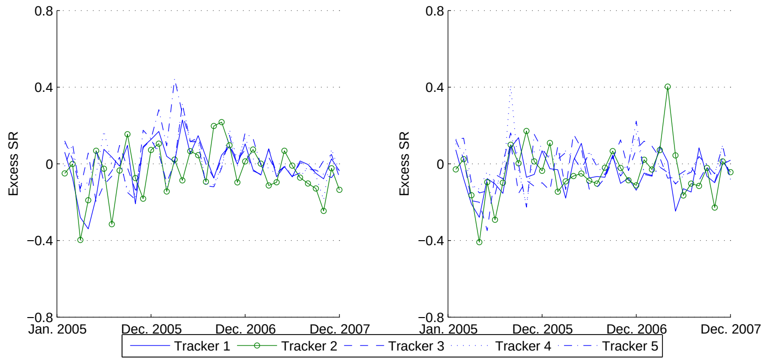
		Replicator 1	Replicator 2	Replicator 3	Replicator 4	Replicator 5
TE Opt.	2005	3	7	3	2	4
	2006	3	8	2	0	0
	2007	2	7	3	0	0
	sum	8	22	8	2	4
Ext. 1	2005	3	9	4	5	7
	2006	5	13	0	0	2
	2007	2	13	1	1	6
	sum	10	35	5	6	15
Ext. 2	2005	3	8	3	4	4
	2006	0	7	0	1	0
	2007	1	9	1	2	1
	sum	4	24	4	7	5

the middle panel. The statistical results suggest that the excess Sharpe ratios are not significantly different from 0 at a 5% confidence level. The insignificancy of the excess Sharpe ratios indicates that the extra risk taken by the trackers has been compensated by return premiums.

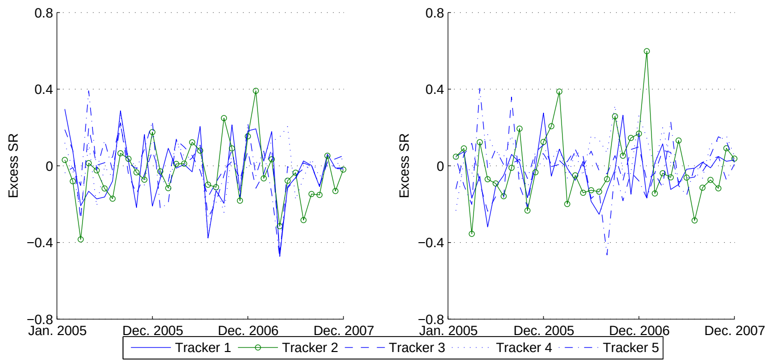
When comparing the left and right panels of Figure 6, the tolerance triggered rebalancing tends to produce more consisting positive deviations of the excess Sharpe ratio, than that using the calendar based rebalancing over the years 2006 and 2007. These positive deviations also explain the reason of the higher β values shown in Table 2.

Cost efficiency is an important criterion for rebalancing strategy evaluation. In the experiments, the rebalancing stage consisted of three years. Thus there would be 12 rebalances if one takes 60 trading days as rebalancing interval in the calendar based strategy. It will be interesting to know how many rebalancing times occur during the rebalancing period while using the tolerance triggered rebalancing; Table 4 records the number of rebalances in this case. From the table, it is found that excepting Replicator 2, most of the rebalancing times of others in the rebalancing stage are less than 12, suggesting the tolerance triggered strategy is more cost-efficient than the calendar based strategy.

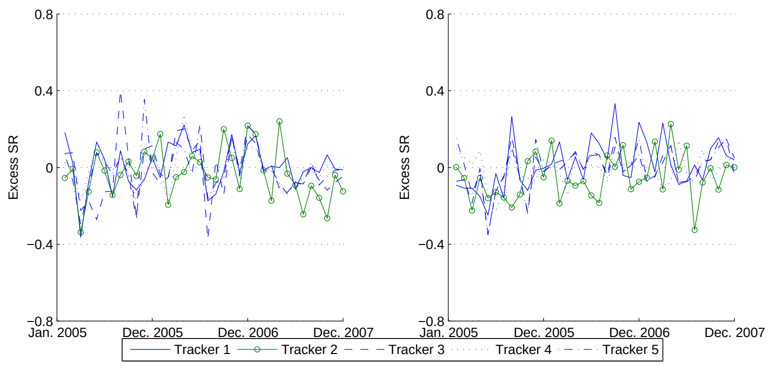
To explore the reason for the high rebalancing times of Replicator 2, the actual fund returns are used to be studied. Table 5 shows the means and standard deviations which were computed based on the actual and replicated returns. It should be noted that Fund 2 exceeds the other four funds by yielding almost two



(a) Tracking Error Optimization



(b) Tracking Error and Excess Return Optimization



(c) Tracking Error and Loss Aversion Optimization

Figure 6: Out-of-Sample excess Sharpe ratios from using the three objective functions and two rebalancing strategies (left – calendar based rebalancing, right – tolerance triggered rebalancing).

and four times more than that of others in 2005 and 2007, but it gains only half of the rewards of others in 2006. The remarkable performance of Fund 2 implies that the fund may adopt a different management strategy from that of the passive management style, thereby resulting in the high tracking errors (the case of solid line with circle marker in Figure 5) and the high rebalancing times.

In Table 5, one may note that most of the negative returns of the replicators occur in 2005; correspondingly, it is straightforward to observe that the excess Sharpe ratios are negative during 2005 from Figure 6. This evidence indicates that the tracker portfolios constructed at the beginning are not robust. One possible explanation for this is that the authors made use of one year observations, i.e. the fist 250 daily data to construct the trackers. Using long historical data, e.g. one year or half year observations to analyze index compositions may be appropriate, whereas it may lead to unreliable outcomes for fund tracking. The main difference between indices and index funds is that the funds are under professional management. To maintain effective diversification of portfolios, fund mangers normally do not keep their holdings unchanged for such long periods, e.g. one year. Therefore, a long data sample will shield the true fund compositions if there is rebalance taken place just before the tracker construction; thus, using short term data, e.g. quarterly data may be more appropriate. Apart from directly using short term data samples, one can modify the TE definition to include a weighting factor, for more recent time periods getting a higher weighting than other time periods (see Beasley et al. [2003]).

4 Conclusion

In this paper, the authors develop a fund tracking model in order to track index mutual funds with constraints on the cardinality size, assets' weights, transaction costs, and integer constraints. For the first time, this paper proposes to decompose the index mutual fund replication into the traditional index tracking problem and a multi-period optimization problem. By employing a heuristic method to solve the optimization problem, an empirical study is performed to track five S&P 500 mutual funds dynamically. The regression results show that the model replicates the first two moments of index fund returns by using limited equities; moreover, the optimized tracker portfolios do not exhibit significant difference between the original and replicated Sharpe ratios. By setting the tracking error tolerance at twice the in-sample tracking error in the tolerance triggered rebalancing, the model produced the same tracking error magnitude as that using the calendar based rebalancing. Also, it has been shown that tolerance triggered rebalancing outperformed the calendar based rebalancing in terms of both tracking ability and cost-efficiency. Financial practitioners may employ the model to build up their

Table 5: Means and Standard Deviations of Actual and Replicated Returns.

		Fund Return		Replicated Return											
				TE Optmization				Ext. 1				Ext. 2			
		Mean(10^{-3})	S.D.(10^{-2})	Mean(10^{-3})		S.D.(10^{-2})		Mean(10^{-3})		S.D.(10^{-2})		Mean(10^{-3})		S.D.(10^{-2})	
	Replications			C.	T.	C.	T.	C.	T.	C.	T.	C.	T.	C.	T.
2005	1	0.18	0.64	-0.44	-0.34	0.75	0.67	-0.06	-0.28	0.66	0.79	-0.15	-0.36	0.73	0.72
	2	0.38	0.76	-0.23	-0.16	0.89	0.88	-0.05	0.21	0.93	0.93	-0.18	-0.44	0.92	0.91
	3	0.18	0.64	-0.06	-0.46	0.69	0.69	0.10	0.17	0.75	0.76	-0.13	-0.24	0.72	0.70
	4	0.19	0.64	-0.18	-0.04	0.66	0.67	-0.03	-0.02	0.77	0.79	-0.03	0.12	0.77	0.74
	5	0.15	0.66	-0.18	0.05	0.73	0.65	0.21	-0.05	0.75	0.73	-0.27	0.09	0.79	0.74
2006	1	0.42	0.63	0.95	0.21	0.63	0.68	0.35	0.34	0.75	0.77	0.63	0.68	0.61	0.66
	2	0.20	0.90	0.45	-0.08	0.91	0.91	0.44	-0.01	0.98	1.00	0.43	0.14	0.98	0.96
	3	0.41	0.62	0.89	0.23	0.60	0.65	0.44	0.15	0.70	0.70	0.70	0.59	0.65	0.60
	4	0.42	0.63	0.84	0.56	0.60	0.59	0.33	-0.01	0.76	0.77	0.55	0.32	0.63	0.64
	5	0.43	0.62	1.11	0.56	0.58	0.59	0.45	0.29	0.79	0.73	0.75	0.58	0.64	0.59
2007	1	0.17	0.98	0.11	-0.18	1.03	0.97	-0.01	0.22	1.00	0.98	0.20	0.51	0.99	1.00
	2	0.71	1.10	0.03	0.39	1.07	1.11	-0.03	0.55	1.29	1.18	0.07	0.40	1.08	1.14
	3	0.18	0.98	0.13	0.05	1.04	0.95	-0.11	0.11	1.01	1.01	0.07	0.43	1.00	0.92
	4	0.17	0.98	0.20	0.07	1.03	0.96	0.33	0.34	1.01	1.04	0.13	0.26	0.95	1.07
	5	0.17	0.98	0.11	0.08	1.01	0.96	-0.16	0.11	1.00	1.03	0.06	0.36	1.00	0.92

own portfolios based on any interesting index mutual funds for certain purposes, such as the funds from different geographical areas for global investments. In future research, the short selling constraint may be relaxed; some financial products, such as bonds, futures and options may be included into tracker portfolios for more advanced applications, e.g. enhanced index funds (EIF) replications. Differential Evolution is reliable and flexible enough for these further extensions.

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